
MEASURED QUANTUM GROUPOIDS

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ABSTRACT. — In this article, part of the author's thesis [Les03], we propose a definition for measured quantum groupoid. The aim is the construction of objects with duality including both quantum groups and groupoids. We base ourselves on J. Kustermans and S. Vaes' works about locally compact quantum groups that we generalize thanks to formalism introduced by M. Enock and J.M. Vallin in the case of inclusion of von Neumann algebras. From a structure of Hopf-bimodule with left and right invariant operator-valued weights, we define a fundamental pseudo-multiplicative unitary. We introduce the notion of quasi-invariant weight on the basis and, then, we construct an antipode with polar decomposition, a coinvolution, a scaling group, a modulus and a scaling operator. This theory is illustrated with different examples. Duality of measured quantum groupoids will be discussed in a forthcoming article.

RÉSUMÉ (*Groupeïdes quantiques mesurés*). — Dans cet article, extrait d'une partie de la thèse [Les03] de l'auteur, on propose une définition des groupeïdes quantiques mesurés. L'objectif est la construction d'objets, munis d'une dualité, qui englobent à la fois les groupeïdes et les groupes quantiques. On s'appuie sur les travaux de J. Kustermans et S. Vaes concernant les groupes quantiques localement compacts qu'on généralise grâce au formalisme introduit par M. Enock et J.M. Vallin à propos des inclusions d'algèbres de von Neumann. À partir d'un bimodule de Hopf muni de poids opératoriels invariants à gauche et à droite, on définit un unitaire pseudo-multiplicatif fondamental. On introduit la notion de poids quasi-invariant sur la base et on construit une antipode avec décomposition polaire, une coinvolution, un groupe d'échelle, un module et un opérateur d'échelle. Cette théorie est illustrée par différents exemples. La dualité de ces objets sera discutée dans un prochain article.

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1. Introduction

1.1. Historique. — Theory of quantum groups has lot of developments in operator algebras setting. Many contributions are given by [KaV74], [Wor88], [ES89], [MN91], [BS93], [Wor95], [Wor96], [VDa98], [KV00]. In particular, J. Kustermans and S. Vaes' work is crucial: in [KV00], they propose a simple definition for locally compact quantum groups which gathers all known examples (locally compact groups, quantum compacts groupe [Wor95], quantum group $ax+b$ [Wor01], [WZ02], Woronowicz' algebra [MN91]...) and they find a general framework for duality of these objects. The very few number of axioms gives the theory a high manageability which is proved with recent developments in many directions (actions of locally compact quantum groups [Vae01b], induced co-representations [Kus02], cocycle bi-crossed products [VV03]). They complete their work with a theory of locally compact quantum groups in the von Neumann setting [KV03].

In geometry, groups are rather defined by their actions. Groupoids category contains groups, group actions and equivalence relation. It is used by G.W Mackey and P. Hahn ([Mac66], [Hah78a] and [Hah78b]), in a measured version, to link theory of groups and ergodic theory. Locally compact groupoids and the operator theory point of view are introduced and studied by J. Renault in [Ren80] and [Ren97]. It covers many interesting examples in differential geometry [Co94] e.g holonomy groupoid of a foliation.

In [Val96], J.M Vallin introduces the notion of Hopf bimodule from which he is able to prove a duality for groupoids. Then, a natural question is to construct a category, containing quantum groups and groupoids, with a duality theory.

In the quantum group case, duality is essentially based on a multiplicative unitary [BS93]. To generalize the notion up to the groupoid case, J.M Vallin introduces pseudo-multiplicative unitaries. In [Val00], he exhibits such an object coming from Hopf bimodule structures for groupoids. Technically speaking, Connes-Sauvageot's theory of relative tensor products is intensively used.

In the case of depth 2 inclusions of von Neumann algebras, M. Enock and J.M Vallin, and then, M. Enock underline two "quantum groupoids" in duality. They also use Hopf bimodules and pseudo-multiplicative unitaries. At this stage, a non trivial modular theory on the basis (the equivalent for units of a groupoid) is revealed to be necessary and a simple generalization of axioms quantum groups is not sufficient to construct quantum groupoids category: we have to add an axiom on the basis [Eno00] i.e we use a special weight to do the construction. The results are improved in [Eno04].

In [Eno02], M. Enock studies in detail pseudo-multiplicative unitaries and introduces an analogous notion of S. Baaj and G. Skandalis' regularity. In quantum groups, the fundamental multiplicative unitary is weakly regular and manageable in the sense of Woronowicz. Such properties have to be satisfied

in quantum groupoids. Moreover, M. Enock defines and studies compact (resp. discrete) quantum groupoids which have to enter into the general theory.

Lot of works have been led about quantum groupoids but essentially in finite dimension. We have to quote weak Hopf C^* -algebras introduced by G. Böhm, F. Nill and K. Szlachányi [BNS99], [BSz96], and then studied by F. Nill and L. Vainerman [Nik02], [Nil98], [NV00], [NV02]. J.M Vallin develops a quantum groupoids theory in finite dimension thanks to multiplicative partial isometries [Val01], [Val02]. He proves that his theory coincide exactly with weak Hopf C^* -algebras.

1.2. Aims and Methods. — In this article, we propose a definition for measured quantum groupoids in any dimensions. "Measured" means we are in the von Neumann setting and we assume existence of the analogous of a measure. We use a similar approach as J. Kustermans and S. Vaes' theory with the formalism of Hopf bimodules and pseudo-multiplicative unitaries. We develop the theory by constructing all fundamental objects and we give some examples. In a forthcoming article, we will study duality within the category.

1.3. Study plan. — After brief recalls about tools and technical points, we define objects we will use. We start by associating a pseudo-multiplicative unitary to every Hopf bimodule with invariant operator-valued weights. This unitary gathers all informations on the structure so that we can re-construct von Neumann algebra and co-product. Then a measured quantum groupoid will be a Hopf bimodule with invariant operator-valued weights which are "adapted" in a certain sense. This hypothesis corresponds to the choice of a special weight on the basis to do the constructions.

Thanks to this axiom, we are able to construct fundamental objects of the structure: first of all, the antipode S , the polar decomposition of which is given by a co-involution R and a one-parameter group of automorphisms called scaling group τ . In particular, we show that S, R and τ are independent of operator-valued weights. Also, we introduce a modulus, which corresponds to modulus of groupoids, and a scaling operator, affiliated to the hyper-center of the Hopf bimodule. They come from Radon-Nikodym's cocycle of right invariant operator-valued weight with respect to left invariant one thanks to proposition 5.2 of [Vae01a]. So, it is the existence of a suitable weight on the basis N which allows us to construct modulus like in the groupoid case with a quasi-invariant measure on $G^{\{0\}}$. Scaling operator is the object which corresponds to scaling factor in locally compact quantum groups. We are also able to prove uniqueness of invariant operator-valued weight up to an element of basis center. Finally, we prove a "manageability" property of the fundamental pseudo-multiplicative unitary.

We have a lot of examples for locally compact quantum groups thanks to Woronowicz [Wor91], [Wor01], [WZ02], [Wor87] and the cocycle bi-crossed

product due to S. Vaes and L. Vainerman [VV03]. Theory of measured quantum groupoids has also a lot of examples: groupoids, weak Hopf C^* -algebras, quantum groups, quantum groupoids of compact (resp. discrete) type... which are characterized in the general theory. Depth 2 inclusions of von Neumann algebras with semi-finite basis also enter into our setting. Finally, we state stability of the category by direct sum (which reflects the stability of groupoids under disjoint unions), finite tensor product and direct integrals. Then, we are able to construct new examples: in particular we can exhibit quantum groupoids with non scalar scaling operator.

2. Recalls

2.1. Weights and operator-valued weights [Str81], [Tak03]. — Let N be a von Neumann and ψ a normal, semi-finite faithful (n.s.f) weight on N ; we denote by \mathcal{N}_ψ , \mathcal{M}_ψ , H_ψ , π_ψ , Λ_ψ , J_ψ , Δ_ψ ... canonical objects of Tomita's theory with respect to (w.r.t) ψ .

DEFINITION 2.1. — Let denote by \mathcal{T}_ψ **Tomita's algebra** w.r.t ψ defined by:

$$\{x \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^* \mid x \text{ analytic w.r.t } \sigma^\psi \text{ such that } \sigma_z^\psi(x) \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^* \text{ for all } z \in \mathbb{C}\}$$

By ([Str81], 2.12), we have the following approximating result:

LEMMA 2.2. — *For all $x \in \mathcal{N}_\psi$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of \mathcal{T}_ψ such that:*

- i) $\|x_n\| \leq \|x\|$ for all $n \in \mathbb{N}$;
- ii) $(x_n)_{n \in \mathbb{N}}$ converges to x in the strong topology;
- iii) $(\Lambda_\psi(x_n))_{n \in \mathbb{N}}$ converges to $\Lambda_\psi(x)$ in the norm topology of H_ψ .

Moreover, if $x \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^*$, then we have:

- iv) $(x_n)_{n \in \mathbb{N}}$ converges to x in the $*$ -strong topology;
- iv) $(\Lambda_\psi(x_n^*))_{n \in \mathbb{N}}$ converges to $\Lambda_\psi(x^*)$ in the norm topology of H_ψ .

Let $N \subset M$ be an inclusion of von Neumann algebras and T a normal, semi-finite, faithful (n.s.f) operator-valued weight from M to N . We put:

$$\mathcal{N}_T = \{x \in M \mid T(x^*x) \in N^+\} \text{ and } \mathcal{M}_T = \mathcal{N}_T^* \mathcal{N}_T$$

We can define a n.s.f weight $\psi \circ T$ on M in a natural way. Let us recall theorem 10.6 of [EN96]:

PROPOSITION 2.3. — *Let $N \subset M$ be an inclusion of von Neumann algebras and T be a normal, semi-finite, faithful (n.s.f) operator-valued weight from M to N and ψ a n.s.f weight on N . Then we have:*

i) for all $x \in \mathcal{N}_T$ and $a \in \mathcal{N}_\psi$, xa belongs to $\mathcal{N}_T \cap \mathcal{N}_{\psi \circ T}$, there exists $\Lambda_T(x) \in \text{Hom}_{N^\circ}(H_\psi, H_{\psi \circ T})$ such that:

$$\Lambda_T(x)\Lambda_\psi(a) = \Lambda_{\psi \circ T}(xa)$$

and Λ_T is a morphism of M - N -bimodules from \mathcal{N}_T to $\text{Hom}_{N^\circ}(H_\psi, H_{\psi \circ T})$;

ii) $\mathcal{N}_T \cap \mathcal{N}_{\psi \circ T}$ is a weakly dense ideal of M and $\Lambda_{\psi \circ T}(\mathcal{N}_T \cap \mathcal{N}_{\psi \circ T})$ is dense in $H_{\psi \circ T}$, $\Lambda_{\psi \circ T}(\mathcal{N}_T \cap \mathcal{N}_{\psi \circ T} \cap \mathcal{N}_T^* \cap \mathcal{N}_{\psi \circ T}^*)$ is a core for $\Delta_{\psi \circ T}^{1/2}$ and $\Lambda_T(\mathcal{N}_T)$ is dense in $\text{Hom}_{N^\circ}(H_\psi, H_{\psi \circ T})$ for the s -topology defined by ([BDH88], 1.3);

iii) for all $x \in \mathcal{N}_T$ and $z \in \mathcal{N}_T \cap \mathcal{N}_{\psi \circ T}$, $T(x^*z)$ belongs to \mathcal{N}_ψ and:

$$\Lambda_T(x)^* \Lambda_{\psi \circ T}(z) = \Lambda_\psi(T(x^*z))$$

iv) for all $x, y \in \mathcal{N}_T$:

$$\Lambda_T(y)^* \Lambda_T(x) = \pi_\psi(T(x^*y)) \text{ and } \|\Lambda_T(x)\| = \|T(x^*x)\|^{1/2}$$

and Λ_T is injective.

Let us also recall lemma 10.12 of [EN96]:

PROPOSITION 2.4. — Let $N \subseteq M$ be an inclusion of von Neumann algebras, T a n.s.f operator-valued weight from M to N , ψ a n.s.f weight on N and $x \in \mathcal{M}_T \cap \mathcal{M}_{\psi \circ T}$. If we put:

$$x_n = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} e^{-nt^2} \sigma_t^{\psi \circ T}(x) dt$$

then x_n belongs to $\mathcal{M}_T \cap \mathcal{M}_{\psi \circ T}$ and is analytic w.r.t $\psi \circ T$. The sequence converges to x and is bounded by $\|x\|$. Moreover, $(\Lambda_{\psi \circ T}(x_n))_{n \in \mathbb{N}}$ converges to $\Lambda_{\psi \circ T}(x)$ and $\sigma_z^{\psi \circ T}(x_n) \in \mathcal{M}_T \cap \mathcal{M}_{\psi \circ T}$ for all $z \in \mathbb{C}$.

DEFINITION 2.5. — The set of $x \in \mathcal{N}_\Phi \cap \mathcal{N}_\Phi^* \cap \mathcal{N}_T \cap \mathcal{N}_T^*$, analytic w.r.t σ^Φ such that $\sigma_z^\Phi(x) \in \mathcal{N}_\Phi \cap \mathcal{N}_\Phi^* \cap \mathcal{N}_T \cap \mathcal{N}_T^*$ for all $z \in \mathbb{C}$ is denoted by \mathcal{T}_Φ and is called **Tomita's algebra** w.r.t $\psi \circ T = \Phi$ and T .

Lemma 2.2 is still satisfied with Tomita's algebra w.r.t Φ and T .

2.2. Spatial theory [Co80], [Sau83b], [Tak03]. — Let α be a normal, non-degenerated representation of N on a Hilbert space H . So, H becomes a left N -module and we write ${}_\alpha H$.

DEFINITION 2.6. — [Co80] An element ξ of ${}_\alpha H$ is said to be bounded w.r.t ψ if there exists $C \in \mathbb{R}^+$ such that, for all $y \in \mathcal{N}_\psi$, we have $\|\alpha(y)\xi\| \leq C\|\Lambda_\psi(y)\|$. The set of **bounded elements** w.r.t ψ is denoted by $D({}_\alpha H, \psi)$.

By [Co80] (lemma 2), $D({}_\alpha H, \psi)$ is dense in H and $\alpha(N)'$ -stable. An element ξ of $D({}_\alpha H, \psi)$ gives rise to a bounded operator $R^{\alpha, \psi}(\xi)$ of $\text{Hom}_N(H_\psi, H)$ such that, for all $y \in \mathcal{N}_\psi$:

$$R^{\alpha, \psi}(\xi)\Lambda_\psi(y) = \alpha(y)\xi$$

For all $\xi, \eta \in D({}_\alpha H, \psi)$, we put:

$$\theta^{\alpha, \psi}(\xi, \eta) = R^{\alpha, \psi}(\xi)R^{\alpha, \psi}(\eta)^* \text{ and } \langle \xi, \eta \rangle_{\alpha, \psi} = R^{\alpha, \psi}(\eta)^* R^{\alpha, \psi}(\xi)^*$$

By [Co80] (lemma 2), the linear span of $\theta^{\alpha, \psi}(\xi, \eta)$ is a weakly dense ideal of ${}_\alpha(N)'$. $\langle \xi, \eta \rangle_{\alpha, \psi}$ belongs to $\pi_\psi(N)' = J_\psi \pi_\psi(N) J_\psi$ which is identified with the opposite von Neumann algebra N^o . The linear span of $\langle \xi, \eta \rangle_{\alpha, \psi}$ is weakly dense in N^o .

By [Co80] (proposition 3), there exists a net $(\eta_i)_{i \in I}$ of $D({}_\alpha H, \psi)$ such that:

$$\sum_{i \in I} \theta^{\alpha, \psi}(\eta_i, \eta_i) = 1$$

Such a net is called a (N, ψ) -**basis** of ${}_\alpha H$. By [EN96] (proposition 2.2), we can choose η_i such that $R^{\alpha, \psi}(\eta_i)$ is a partial isometry with two-by-two orthogonal final supports and such that $\langle \eta_i, \eta_j \rangle_{\alpha, \psi} = 0$ unless $i = j$. In the following, we assume these properties hold for all (N, ψ) -basis of ${}_\alpha H$.

Now, let β be a normal, non-degenerated anti-representation from N on H . So H becomes a right N -module and we write H_β . But β is also a representation of N^o . If ψ^o is the n.s.f weight on N^o coming from ψ then $\mathcal{N}_{\psi^o} = \mathcal{N}_\psi^*$ and we identify H_{ψ^o} with H_ψ thanks to:

$$(\Lambda_{\psi^o}(x^*) \mapsto J_\psi \Lambda_\psi(x))$$

DEFINITION 2.7. — [Co80] An element ξ of H_β is said to be bounded w.r.t ψ^o if there exists $C \in \mathbb{R}^+$ such that, for all $y \in \mathcal{N}_\psi$, we have $\|\beta(y^*)\xi\| \leq C\|\Lambda_\psi(y)\|$. The set of **bounded elements** w.r.t ψ^o is denoted by $D(H_\beta, \psi^o)$.

$D({}_\alpha H, \psi)$ is dense in H and $\beta(N)'$ -stable. An element ξ of $D(H_\beta, \psi^o)$ gives rise to a bounded operator $R^{\beta, \psi^o}(\xi)$ of $\text{Hom}_{N^o}(H_\psi, H)$ such that, for all $y \in \mathcal{N}_\psi$:

$$R^{\beta, \psi^o}(\xi) \Lambda_\psi(y) = \beta(y^*)\xi$$

For all $\xi, \eta \in D(H_\beta, \psi^o)$, we put:

$$\theta^{\beta, \psi^o}(\xi, \eta) = R^{\beta, \psi^o}(\xi)R^{\beta, \psi^o}(\eta)^* \text{ and } \langle \xi, \eta \rangle_{\beta, \psi^o} = R^{\beta, \psi^o}(\eta)^* R^{\beta, \psi^o}(\xi)^*$$

The linear span of $\theta^{\beta, \psi^o}(\xi, \eta)$ is a weakly dense ideal of $\beta(N)'$. $\langle \xi, \eta \rangle_{\beta, \psi^o}$ belongs to $\pi_\psi(N)$ which is identified with N . The linear span of $\langle \xi, \eta \rangle_{\beta, \psi^o}$ is weakly dense in N . In fact, we know that $\langle \xi, \eta \rangle_{\beta, \psi^o} \in \mathcal{M}_\psi$ by [Co80] (lemma 4) and by [Sau83b] (lemma 1.5), we have

$$\Lambda_\psi(\langle \xi, \eta \rangle_{\beta, \psi^o}) = R^{\beta, \psi^o}(\eta)^* \xi$$

A net $(\xi_i)_{i \in I}$ of ψ^o -bounded elements of H_β is said to be a (N^o, ψ^o) -basis of H_β if:

$$\sum_{i \in I} \theta^{\beta, \psi^o}(\xi_i, \xi_i) = 1$$

and if ξ_i such that $R^{\beta, \psi^o}(\xi_i)$ is a partial isometry with two-by-two orthogonal final supports and such that $\langle \xi_i, \xi_j \rangle_{\alpha, \psi} = 0$ unless $i = j$. Therefore, we have:

$$R^{\beta, \psi^o}(\xi_i) = \theta^{\beta, \psi^o}(\xi_i, \xi_i) R^{\beta, \psi^o}(\xi_i) = R^{\beta, \psi^o}(\xi_i) \langle \xi_i, \xi_i \rangle_{\beta, \psi^o}$$

and, for all $\xi \in D(H_\beta, \psi^o)$:

$$\xi = \sum_{i \in I} R^{\beta, \psi^o}(\xi_i) \Lambda_\psi(\langle \xi, \xi_i \rangle_{\beta, \psi^o})$$

PROPOSITION 2.8. — ([Eno02], proposition 2.10) Let $N \subseteq M$ be an inclusion of von Neumann algebras and T be a n.s.f operator-valued weight from M to N . There exists a net $(e_i)_{i \in I}$ of $\mathcal{N}_T \cap \mathcal{N}_T^* \cap \mathcal{N}_{\psi \circ T} \cap \mathcal{N}_{\psi \circ T}^*$ such that $\Lambda_T(e_i)$ is a partial isometry, $T(e_j^* e_i) = 0$ unless $i = j$ and with orthogonal final supports of sum 1. Moreover, we have $e_i = e_i T(e_i^* e_i)$ for all $i \in I$, and, for all $x \in \mathcal{N}_T$:

$$\Lambda_T(x) = \sum_{i \in I} \Lambda_T(e_i) T(e_i^* x) \quad \text{and} \quad x = \sum_{i \in I} e_i T(e_i^* x)$$

in the weak topology. Such a net is called a basis for (T, ψ^o) . Finally, the net $(\Lambda_{\psi \circ T}(e_i))_{i \in I}$ is a (N^o, ψ^o) -basis of $(H_{\psi \circ T})_s$ where s is the anti-representation which sends $y \in N$ to $J_{\psi \circ T} y^* J_{\psi \circ T}$.

2.3. Relative tensor product [Co80], [Sau83b], [Tak03]. — Let H and K be Hilbert space. Let α (resp. β) be a normal and non-degenerated (resp. anti-) representation of N on K (resp. H). Let ψ be a n.s.f weight on N . Following [Sau83b], we put on $D(H_\beta, \psi^o) \odot K$ a scalar product defined by:

$$(\xi_1 \odot \eta_1 | \xi_2 \odot \eta_2) = (\alpha(\langle \xi_1, \xi_2 \rangle_{\beta, \psi^o}) \eta_1 | \eta_2)$$

for all $\xi_1, \xi_2 \in D(H_\beta, \psi^o)$ and $\eta_1, \eta_2 \in K$. We have identified $\pi_\psi(N)$ with N .

DEFINITION 2.9. — The completion of $D(H_\beta, \psi^o) \odot K$ is called **relative tensor product** and is denoted by $H_{\beta \otimes_\alpha K}^\psi$.

The image of $\xi \odot \eta$ in $H_{\beta \otimes_\alpha K}^\psi$ is denoted by $\xi_{\beta \otimes_\alpha K}^\psi \eta$. One should bear in mind that, if we start from another n.s.f weight ψ' on N , we get another Hilbert space which is canonically isomorphic to $H_{\beta \otimes_\alpha K}^\psi$ by ([Sau83b], proposition 2.6). However this isomorphism does not send $\xi_{\beta \otimes_\alpha K}^\psi \eta$ on $\xi_{\beta \otimes_\alpha K}^{\psi'} \eta$.

By [Sau83b] (definition 2.1), relative tensor product can be defined from the scalar product:

$$(\xi_1 \odot \eta_1 | \xi_2 \odot \eta_2) = (\beta(\langle \eta_1, \eta_2 \rangle_{\alpha, \psi}) \xi_1 | \xi_2)$$

for all $\xi_1, \xi_2 \in H$ and $\eta_1, \eta_2 \in D({}_\alpha K, \psi)$ that's why we can define a one-to-one flip from $H \underset{\psi}{\beta} \otimes_\alpha K$ onto $K \underset{\psi^o}{\alpha} \otimes_\beta H$ such that:

$$\sigma_\psi(\xi \underset{\psi}{\beta} \otimes_\alpha \eta) = \eta \underset{\psi^o}{\alpha} \otimes_\beta \xi$$

for all $\xi \in D(H_\beta, \psi)$ (resp. $\xi \in H$) and $\eta \in K$ (resp. $\eta \in D({}_\alpha K, \psi)$). The flip gives rise at the operator level to ς_ψ from $\mathcal{L}(H \underset{\psi}{\beta} \otimes_\alpha K)$ onto $\mathcal{L}(K \underset{\psi^o}{\alpha} \otimes_\beta H)$

such that:

$$\varsigma_\psi(X) = \sigma_\psi X \sigma_\psi^*$$

Canonical isomorphisms of change of weights send ς_ψ on $\varsigma_{\psi'}$ so that we write ς_N without any reference to the weight on N .

For all $\xi \in D(H_\beta, \psi^o)$ and $\eta \in D({}_\alpha K, \psi)$, we define bounded operators:

$$\begin{aligned} \lambda_\xi^{\beta, \alpha} : K &\rightarrow H \underset{\psi}{\beta} \otimes_\alpha K & \text{and} & \quad \rho_\eta^{\beta, \alpha} : H \rightarrow H \underset{\psi}{\beta} \otimes_\alpha K \\ \eta &\mapsto \xi \underset{\psi}{\beta} \otimes_\alpha \eta & & \quad \xi \mapsto \xi \underset{\psi}{\beta} \otimes_\alpha \eta \end{aligned}$$

Then, we have:

$$(\lambda_\xi^{\beta, \alpha})^* \lambda_\xi^{\beta, \alpha} = \alpha(< \xi, \xi >_{\beta, \psi^o}) \text{ and } (\rho_\eta^{\beta, \alpha})^* \rho_\eta^{\beta, \alpha} = \beta(< \eta, \eta >_{\alpha, \psi})$$

By [Sau83b] (remark 2.2), we know that $D({}_\alpha K, \psi)$ is $\alpha(\sigma_{-i/2}^\psi(\mathcal{D}(\sigma_{-i/2}^\psi)))$ -stable and for all $\xi \in H$, $\eta \in D({}_\alpha K, \psi)$ and $y \in \mathcal{D}(\sigma_{-i/2}^\psi)$, we have:

$$\beta(y) \xi \underset{\psi}{\beta} \otimes_\alpha \eta = \xi \underset{\psi}{\beta} \otimes_\alpha \alpha(\sigma_{-i/2}^\psi(y)) \eta$$

LEMMA 2.10. — *If $\xi' \underset{\psi}{\beta} \otimes_\alpha \eta = 0$ for all $\xi' \in D(H_\beta, \psi^o)$ then $\eta = 0$.*

Proof. — For all $\xi, \xi' \in D(H_\beta, \psi^o)$, we have:

$$\alpha(< \xi', \xi >_{\beta, \psi^o}) \eta = (\lambda_\xi^{\beta, \alpha})^* \lambda_{\xi'}^{\beta, \alpha} \eta = (\lambda_\xi^{\beta, \alpha})^* (\xi' \underset{\psi}{\beta} \otimes_\alpha \eta) = 0$$

Since the linear span of $< \xi', \xi >_{\beta, \psi^o}$ is dense in N , we get $\eta = 0$. \square

PROPOSITION 2.11. — *Assume $H \neq \{0\}$. Let K' be a closed subspace of K such that $\alpha(N)K' \subseteq K'$. Then:*

$$H \underset{\psi}{\beta} \otimes_\alpha K = H \underset{\psi}{\beta} \otimes_\alpha K' \quad \Rightarrow \quad K = K'$$

Proof. — Let $\eta \in K'^\perp$. For all $\xi, \xi' \in D(H_\beta, \psi^o)$ and $k \in K'$, we have:

$$(\xi \underset{\psi}{\beta} \otimes_\alpha k | \xi' \underset{\psi}{\beta} \otimes_\alpha \eta) = (\alpha(< \xi, \xi' >_{\beta, \psi^o}) k | \eta) = 0$$

Therefore, for all $\xi' \in D(H_\beta, \psi^o)$, we have:

$$\xi' \underset{\psi}{\beta \otimes_\alpha} \eta \in (H \underset{\psi}{\beta \otimes_\alpha} K')^\perp = (H \underset{\psi}{\beta \otimes_\alpha} K)^\perp = \{0\}$$

By the previous lemma, we get $\eta = 0$ and $K = K'$. \square

Let $x \in \beta(N)' \cap \mathcal{L}(H)$ and $y \in \alpha(N)' \cap \mathcal{L}(K)$. By [Sau83a], 2.3 and 2.6, we can naturally define an operator $x \underset{\psi}{\beta \otimes_\alpha} y$ on $H \underset{\psi}{\beta \otimes_\alpha} K$. Canonical isomorphism of change of weights sends $x \underset{\psi}{\beta \otimes_\alpha} y$ on $x \underset{\psi'}{\beta \otimes_\alpha} y$ so that we write $x \underset{N}{\beta \otimes_\alpha} y$ without any reference to the weight.

Let P be a von Neumann algebra and ϵ a normal and non-degenerated anti-representation of P on K such that $\epsilon(P)' \subseteq \alpha(N)$. K is equipped with a $N - P$ -bimodule structure denoted by $\underset{\alpha}{K}_\epsilon$. For all $y \in P$, $1_H \underset{\psi}{\beta \otimes_\alpha} \epsilon(y)$ is an operator on $H \underset{\psi}{\beta \otimes_\alpha} K$ so that we define a representation of P on $H \underset{\psi}{\beta \otimes_\alpha} K$ still denoted by ϵ . If H is a $Q - N$ -bimodule, then $H \underset{\psi}{\beta \otimes_\alpha} K$ becomes a $Q - P$ -bimodule (Connes' fusion of bimodules). If ν is a n.s.f weight on P and $\underset{\zeta}{L}$ a left P -module. It is possible to define two Hilbert spaces $(H \underset{\psi}{\beta \otimes_\alpha} K) \underset{\nu}{\epsilon \otimes_\zeta} L$ and $H \underset{\psi}{\beta \otimes_\alpha} (K \underset{\nu}{\epsilon \otimes_\zeta} L)$. These two $\beta(N)' - \zeta(P)'^o$ -bimodules are isomorphic. (The proof of [Val96], lemme 2.1.3, in the case of commutative $N = P$ is still valid). We speak about associativity of relative tensor product and we write $H \underset{\psi}{\beta \otimes_\alpha} K \underset{\nu}{\epsilon \otimes_\zeta} L$ without parenthesis.

We identify $H \underset{\psi}{\beta \otimes_\alpha} K$ and K as left N -modules by $\Lambda_\psi(y) \underset{\psi}{\beta \otimes_\alpha} \eta \mapsto \alpha(y)\eta$ for all $y \in \mathcal{N}_\psi$. By [EN96], 3.10, we have:

$$\lambda_\xi^{\beta, \alpha} = R^{\beta, \psi^o}(\xi) \underset{\psi}{\beta \otimes_\alpha} 1_K$$

We recall proposition 2.3 of [Eno02]:

PROPOSITION 2.12. — *Let $(\xi_i)_{i \in I}$ be a (N^o, ψ^o) -basis of H_β . Then:*

i) for all $\xi \in D(H_\beta, \psi^o)$ and $\eta \in K$, we have:

$$\xi \underset{\psi}{\beta \otimes_\alpha} \eta = \sum_{i \in I} \xi_i \underset{\psi}{\beta \otimes_\alpha} \alpha(< \xi, \xi_i >_{\beta, \psi^o}) \eta$$

ii) we have the following decomposition:

$$H \underset{\psi}{\beta \otimes_\alpha} K = \bigoplus_{i \in I} (\xi_i \underset{\psi}{\beta \otimes_\alpha} \alpha(< \xi_i, \xi_i >_{\beta, \psi^o}) K)$$

To end the paragraph, we detail finite dimension case. We assume that N , H and K are of finite dimensions. $H \underset{\psi}{\beta \otimes \alpha} K$ can be identified with a subspace of $H \otimes K$. We denote by Tr the normalized canonical trace on K and $\tau = \text{Tr} \circ \alpha$. There exist a projection $e_{\beta, \alpha} \in \beta(N) \otimes \alpha(N)$ and $n_o \in Z(N)^+$ such that $(\text{id} \otimes \text{Tr})(e_{\beta, \alpha}) = \beta(n_o)$. Let d be the Radon-Nikodym derivative of ψ w.r.t τ . By [EV00], 2.4, and proposition 2.7 of [Sau83b], for all $\xi, \eta \in H$:

$$I_{\beta, \alpha}^{\psi} : \xi \underset{\psi}{\beta \otimes \alpha} \eta \mapsto \xi \underset{\tau}{\beta \otimes \alpha} \alpha(d)^{1/2} \eta \mapsto e_{\beta, \alpha} (\beta(n_o))^{-1/2} \xi \otimes \alpha(d)^{1/2} \eta$$

defines an isometric isomorphism of $\beta(N)' - \alpha(N)^{o}$ -bimodules from $H \underset{\psi}{\beta \otimes \alpha} K$ onto a subspace of $H \otimes K$, the final support of which is $e_{\beta, \alpha}$.

2.4. Fiber product [Val96], [EV00]. — We use previous notations. Let M_1 (resp. M_2) be a von Neumann algebra on H (resp. K) such that $\beta(N) \subseteq M_1$ (resp. $\alpha(N) \subseteq M_2$). We denote by $M'_1 \underset{N}{\beta \otimes \alpha} M'_2$ the von Neumann algebra generated by $x \underset{N}{\beta \otimes \alpha} y$ with $x \in M'_1$ and $y \in M'_2$.

DEFINITION 2.13. — The commutant of $M'_1 \underset{N}{\beta \otimes \alpha} M'_2$ in $\mathcal{L}(H \underset{\psi}{\beta \otimes \alpha} K)$ is denoted by $M_1 \underset{N}{\beta \star \alpha} M_2$ and is called **fiber product**.

If P_1 and P_2 are von Neumann algebras like M_1 and M_2 , we have:

- i) $(M_1 \underset{N}{\beta \star \alpha} M_2) \cap (P_1 \underset{N}{\beta \star \alpha} P_2) = (M_1 \cap P_1) \underset{N}{\beta \star \alpha} (M_2 \cap P_2)$
- ii) $\varsigma_N(M_1 \underset{N}{\beta \star \alpha} M_2) = M_2 \underset{N^o}{\alpha \star \beta} M_1$
- iii) $(M_1 \cap \beta(N)') \underset{N}{\beta \otimes \alpha} (M_2 \cap \alpha(N)') \subseteq M_1 \underset{N}{\beta \star \alpha} M_2$
- iv) $M_1 \underset{N}{\beta \star \alpha} \alpha(N) = (M_1 \cap \beta(N)') \underset{N}{\beta \otimes \alpha} 1$

More generally, if β (resp. α) is a normal, non-degenerated $*$ -anti-homomorphism (resp. homomorphism) from N to a von Neumann algebra M_1 (resp. M_2), it is possible to define a von Neumann algebra $M_1 \underset{N}{\beta \star \alpha} M_2$

without any reference to a specific Hilbert space. If P_1 , P_2 , α' and β' are like M_1 , M_2 , α and β and if Φ (resp. Ψ) is a normal $*$ -homomorphism from M_1 (resp. M_2) to P_1 (resp. P_2) such that $\Phi \circ \beta = \beta'$ (resp. $\Psi \circ \alpha = \alpha'$), then we define a normal $*$ -homomorphism by [Sau83a], 1.2.4:

$$\Phi \underset{N}{\beta \star \alpha} \Psi : M_1 \underset{N}{\beta \star \alpha} M_2 \rightarrow P_1 \underset{N}{\beta' \star \alpha'} P_2$$

Assume ${}_{\alpha}K_{\epsilon}$ is a $N - P^o$ -bimodule and ${}_{\zeta}L$ a left P -module. If $\alpha(N) \subseteq M_2$, $\epsilon(P) \subseteq M_2$ and if $\zeta(P) \subseteq M_3$ where M_3 is a von Neumann algebrasur on L ,

then we can construct $M_1 \underset{N}{\beta \star \alpha} (M_2 \underset{N}{\epsilon \star \zeta} M_3)$ and $(M_1 \underset{N}{\beta \star \alpha} M_2) \underset{N}{\epsilon \star \zeta} M_3$. Associativity of relative tensor product induces an isomorphism between these fiber products and we write $M_1 \underset{N}{\beta \star \alpha} M_2 \underset{N}{\epsilon \star \zeta} M_3$ without parenthesis.

Finally, if M_1 and M_2 are of finite dimensions, then we have:

$$M'_1 \underset{N}{\beta \otimes \alpha} M'_2 = (I_{\beta, \alpha}^\psi)^*(M'_1 \otimes M'_2) I_{\beta, \alpha}^\psi \text{ and } M_1 \underset{N}{\beta \star \alpha} M_2 = (I_{\beta, \alpha}^\psi)^*(M_1 \otimes M_2) I_{\beta, \alpha}^\psi$$

Therefore the fiber product can be identified with a reduction of $M_1 \otimes M_2$ by $e_{\beta, \alpha}$ by [EV00], 2.4.

2.5. Slice map [Eno00]. —

2.5.1. For normal forms. — Let $A \in M_1 \underset{N}{\beta \star \alpha} M_2$ and $\xi_1, \xi_2 \in D(H_\beta, \psi^o)$. We define an element of M_2 by:

$$(\omega_{\xi_1, \xi_2} \underset{\psi}{\beta \star \alpha} id)(A) = (\lambda_{\xi_2}^{\beta, \alpha})^* A \lambda_{\xi_1}^{\beta, \alpha}$$

so that we have $((\omega_{\xi_1, \xi_2} \underset{\psi}{\beta \star \alpha} id)(A) \eta_1 | \eta_2) = (A(\xi_1 \underset{\psi}{\beta \otimes \alpha} \eta_1) | \xi_2 \underset{\psi}{\beta \otimes \alpha} \eta_2)$ for all $\eta_1, \eta_2 \in K$. Also, we define an operator of M_1 by:

$$(id \underset{\psi}{\beta \star \alpha} \omega_{\eta_1, \eta_2})(A) = (\rho_{\eta_2}^{\beta, \alpha})^* A \rho_{\eta_1}^{\beta, \alpha}$$

for all $\eta_1, \eta_2 \in D({}_\alpha K, \psi)$. We have a Fubini's formula:

$$\omega_{\eta_1, \eta_2}((\omega_{\xi_1, \xi_2} \underset{\psi}{\beta \star \alpha} id)(A)) = \omega_{\xi_1, \xi_2}((id \underset{\psi}{\beta \star \alpha} \omega_{\eta_1, \eta_2})(A))$$

for all $\xi_1, \xi_2 \in D(H_\beta, \psi^o)$ and $\eta_1, \eta_2 \in D({}_\alpha K, \psi)$.

Equivalently, by ([Eno00], proposition 3.3), for all $\omega_1 \in M_{1*}^+$ and $k_1 \in \mathbb{R}^+$ such that $\omega_1 \circ \beta \leq k_1 \psi$ and for all $\omega_2 \in M_{2*}^+$ and $k_2 \in \mathbb{R}^+$ such $\omega_2 \circ \alpha \leq k_2 \psi$, we have:

$$\omega_2((\omega_1 \underset{\psi}{\beta \star \alpha} id)(A)) = \omega_1((id \underset{\psi}{\beta \star \alpha} \omega_2)(A))$$

2.5.2. For conditional expectations. — If P_2 is a von Neumann algebra such that $\alpha(N) \subseteq P_2 \subseteq M_2$ and if E is a normal, faithful conditional expectation from M_2 onto P_2 , we can define a normal, faithful conditional expectation $(id \underset{N}{\beta \star \alpha} E)$ from $M_1 \underset{N}{\beta \star \alpha} M_2$ onto $M_1 \underset{N}{\beta \star \alpha} P_2$ such that:

$$(\omega \underset{\psi}{\beta \star \alpha} id)(id \underset{N}{\beta \star \alpha} E)(A) = E((\omega \underset{\psi}{\beta \star \alpha} id)(A))$$

for all $A \in M_1 \underset{N}{\beta \star \alpha} M_2$, $\omega \in M_{1*}^+$ and $k_1 \in \mathbb{R}^+$ such that $\omega \circ \beta \leq k_1 \psi$.

2.5.3. *For weights.* — If ϕ_1 is n.s.f weight on M_1 and if A is a positive element of $M_1 \underset{N}{\beta\star_\alpha} M_2$, we can define an element of the extended positive part of M_2 , denoted by $(\phi_1 \underset{\psi}{\beta\star_\alpha} id)(A)$, such that, for all $\eta \in D({}_\alpha L^2(M_2), \psi)$, we have:

$$\|((\phi_1 \underset{\psi}{\beta\star_\alpha} id)(A))^{1/2}\eta\|^2 = \phi_1((id \underset{\psi}{\beta\star_\alpha} \omega_\eta)(A))$$

Moreover, if ϕ_2 is a n.s.f weight on M_2 , we have:

$$\phi_2((\phi_1 \underset{\psi}{\beta\star_\alpha} id)(A)) = \phi_1((id \underset{\psi}{\beta\star_\alpha} \phi_2)(A))$$

Let $(\omega_i)_{i \in I}$ be an increasing net of normal forms such that $\phi_1 = \sup_{i \in I} \omega_i$. Then we have $(\phi_1 \underset{\psi}{\beta\star_\alpha} id)(A) = \sup_i (\omega_i \underset{\psi}{\beta\star_\alpha} id)(A)$.

2.5.4. *For operator-valued weights.* — Let P_1 be a von Neumann algebra such that $\beta(N) \subseteq P_1 \subseteq M_1$ and Φ_i ($i = 1, 2$) be operator-valued n.s.f weights from M_i to P_i . By [Eno00], for all positive operator $A \in M_1 \underset{N}{\beta\star_\alpha} M_2$, there exists an element $(\Phi_1 \underset{N}{\beta\star_\alpha} id)(A)$ belonging to $P_1 \underset{N}{\beta\star_\alpha} M_2$ such that, for all $\xi \in L^2(P_1)$ and $\eta \in D({}_\alpha K, \psi)$, we have:

$$\|((\Phi_1 \underset{N}{\beta\star_\alpha} id)(A))^{1/2}(\xi \underset{\psi}{\beta\otimes_\alpha} \eta)\|^2 = \|[\Phi_1((id \underset{\psi}{\beta\star_\alpha} \omega_{\eta, \eta})(A))]^{1/2}\xi\|^2$$

If ϕ_1 is a n.s.f weight on P_1 , we have:

$$(\phi_1 \circ \Phi_1 \underset{N}{\beta\star_\alpha} id)(A) = (\phi_1 \underset{\psi}{\beta\star_\alpha} id)(\Phi_1 \underset{N}{\beta\star_\alpha} id)(A)$$

Also, we define an element $(id \underset{N}{\beta\star_\alpha} \Phi_2)(A)$ of the extended positive part of $M_1 \underset{N}{\beta\star_\alpha} P_2$ and we have:

$$(id \underset{N}{\beta\star_\alpha} \Phi_2)((\Phi_1 \underset{N}{\beta\star_\alpha} id)(A)) = (\Phi_1 \underset{N}{\beta\star_\alpha} id)((id \underset{N}{\beta\star_\alpha} \Phi_2)(A))$$

REMARK 2.14. — We have seen that we can identify $M_1 \underset{N}{\beta\star_\alpha} \alpha(N)$ with $M_1 \cap \beta(N)'$. Then, it is easy to check that the slice map $id \underset{\psi}{\beta\star_\alpha} \psi \circ \alpha^{-1}$ (if α is injective) is just the injection of $M_1 \underset{N}{\beta\star_\alpha} \alpha(N)$ into M_1 . Also we see on that example that, if ϕ_1 is a n.s.f weight on M_1 , then $\phi_1 \underset{N}{\beta\star_\alpha} id$ (which is equal to $\phi_1|_{M_1 \cap \beta(N)'}$) needs not to be semi-finite.

3. Fundamental pseudo-multiplicative unitary

In this section, we construct a fundamental pseudo-multiplicative unitary from a Hopf bimodule with a left invariant operator-valued weight and a right invariant operator-valued weight. Let N and M be von Neumann algebras, α (resp. β) be a faithful, non-degenerate, normal (resp. anti-) representation from N to M . We suppose that $\alpha(N) \subseteq \beta(N)'$.

3.1. Definitions. —

DEFINITION 3.1. — A quintuplet $(N, M, \alpha, \beta, \Gamma)$ is said to be a **Hopf bimodule** of basis N if Γ is a normal $*$ -homomorphism from M into $M \underset{N}{\beta \star \alpha} M$ such that, for all $n, m \in N$, we have:

- i) $\Gamma(\alpha(n)\beta(m)) = \alpha(n) \underset{N}{\beta \otimes \alpha} \beta(m)$
- ii) Γ is co-associative: $(\Gamma \underset{N}{\beta \star \alpha} id) \circ \Gamma = (id \underset{N}{\beta \star \alpha} \Gamma) \circ \Gamma$

One should notice that property i) is necessary in order to write down the formula given in ii). $(N^o, M, \beta, \alpha, \varsigma_N \circ \Gamma)$ is a Hopf bimodule called opposite Hopf bimodule. If N is commutative, $\alpha = \beta$ and $\Gamma = \varsigma_N \circ \Gamma$, then $(N, M, \alpha, \alpha, \Gamma)$ is equal to its opposite: we shall speak about a symmetric Hopf bimodule.

DEFINITION 3.2. — Let $(N, M, \alpha, \beta, \Gamma)$ be a Hopf bimodule. A normal, semi-finite, faithful operator-valued weight from M to $\alpha(N)$ is said to be **left invariant** if:

$$(id \underset{N}{\beta \star \alpha} T_L)\Gamma(x) = T_L(x) \underset{N}{\beta \otimes \alpha} 1 \quad \text{for all } x \in \mathcal{M}_{T_L}^+$$

In the same way, a normal, semi-finite, faithful operator-valued weight from M to $\beta(N)$ is said to be **right invariant** if:

$$(T_R \underset{N}{\beta \star \alpha} id)\Gamma(x) = 1 \underset{N}{\beta \otimes \alpha} T_R(x) \quad \text{for all } x \in \mathcal{M}_{T_R}^+$$

We give several examples in the last section. In this section, $(N, M, \alpha, \beta, \Gamma)$ is a Hopf bimodule with a left operator-valued weight T_L and a right operator-valued weight T_R .

DEFINITION 3.3. — A $*$ -anti-automorphism R of M is said to be a **co-involution** if $R \circ \alpha = \beta$, $R^2 = id$ and $\varsigma_{N^o} \circ (R \underset{N}{\beta \star \alpha} R) \circ \Gamma = \Gamma \circ R$.

REMARK 3.4. — With the previous notations, let us notice that $R \circ T_L \circ R$ is a right invariant operator-valued weight from M to $\beta(N)$. Also, let us say that R is an anti-isomorphism of Hopf bimodule from the bimodule and its symmetric.

Let μ be a normal, semi-finite, faithful weight of N . We put:

$$\Phi = \mu \circ \alpha^{-1} \circ T_L \text{ and } \Psi = \mu \circ \beta^{-1} \circ T_R$$

so that, for all $x \in M^+$, we have:

$$(id \underset{\mu}{\beta} \star_{\alpha} \Phi) \Gamma(x) = T_L(x) \text{ and } (\Psi \underset{\mu}{\beta} \star_{\alpha} id) \Gamma(x) = T_R(x)$$

If H denote a Hilbert space on which M acts, then N acts on H , also, by way of α and β . We shall denote again α (resp. β) for (resp. anti-) the representation of N on H .

3.2. Construction of the fundamental isometry. —

DEFINITION 3.5. — Let define $\hat{\beta}$ and $\hat{\alpha}$ by:

$$\begin{aligned} \hat{\beta} : N &\rightarrow \mathcal{L}(H_{\Phi}) & \text{and} & & \hat{\alpha} : N &\rightarrow \mathcal{L}(H_{\Psi}) \\ x &\mapsto J_{\Phi} \alpha(x^*) J_{\Phi} & & & x &\mapsto J_{\Psi} \beta(x^*) J_{\Psi} \end{aligned}$$

Then $\hat{\beta}$ (resp. $\hat{\alpha}$) is a normal, non-degenerate and faithful anti-representation (resp. representation) from N to $\mathcal{L}(H_{\Phi})$ (resp. $\mathcal{L}(H_{\Psi})$).

PROPOSITION 3.6. — We have $\Lambda_{\Phi}(\mathcal{N}_{T_L} \cap \mathcal{N}_{\Phi}) \subseteq D((H_{\Phi})_{\hat{\beta}}, \mu^o)$ and for all $a \in \mathcal{N}_{T_L} \cap \mathcal{N}_{\Phi}$, we have:

$$R^{\hat{\beta}, \mu^o}(\Lambda_{\Phi}(a)) = \Lambda_{T_L}(a)$$

Also, we have $\Lambda_{\Psi}(\mathcal{N}_{T_R} \cap \mathcal{N}_{\Psi}) \subseteq D((H_{\Psi})_{\hat{\alpha}}, \mu)$ and for all $b \in \mathcal{N}_{T_R} \cap \mathcal{N}_{\Psi}$, then:

$$R^{\hat{\alpha}, \mu}(\Lambda_{\Psi}(b)) = \Lambda_{T_R}(b)$$

REMARK 3.7. — We identify H_{μ} with $H_{\mu \circ \alpha^{-1}}$ and H_{μ} with $H_{\mu \circ \beta^{-1}}$.

Proof. — Let $y \in \mathcal{N}_{\mu}$ analytic w.r.t μ . We have:

$$\begin{aligned} \hat{\beta}(y^*) \Lambda_{\Phi}(a) &= \Lambda_{\Phi}(a \sigma_{-i/2}^{\Phi}(\alpha(y^*))) = \Lambda_{\Phi}(a \sigma_{-i/2}^{\mu \circ \alpha^{-1}}(\alpha(y^*))) \\ &= \Lambda_{\Phi}(a \alpha(\sigma_{-i/2}^{\mu}(y^*))) = \Lambda_{T_L}(a) \Lambda_{\mu}(\sigma_{-i/2}^{\mu}(y^*)) = \Lambda_{T_L}(a) J_{\mu} \Lambda_{\mu}(y) \end{aligned}$$

Thanks to lemma 2.2, we get $\hat{\beta}(y^*) \Lambda_{\Phi}(a) = \Lambda_{T_L}(a) J_{\mu} \Lambda_{\mu}(y)$, for all $y \in \mathcal{N}_{\mu}$, which gives the first part of the proposition. The end of the proof is very similar. \square

PROPOSITION 3.8. — We have $J_{\Phi} D((H_{\Phi})_{\hat{\beta}}, \mu^o) = D((H_{\Phi})_{\beta}, \mu)$ and for all $\eta \in D((H_{\Phi})_{\hat{\beta}}, \mu^o)$, we have:

$$R^{\alpha, \mu}(J_{\Phi} \eta) = J_{\Phi} R^{\hat{\beta}, \mu^o}(\eta) J_{\mu}$$

Also, we have $J_{\Psi} D((H_{\Psi})_{\hat{\alpha}}, \mu) = D((H_{\Psi})_{\alpha}, \mu^o)$ and for all $\xi \in D((H_{\Psi})_{\hat{\alpha}}, \mu)$, we have:

$$R^{\beta, \mu^o}(J_{\Psi} \xi) = J_{\Psi} R^{\hat{\alpha}, \mu}(\xi) J_{\mu}$$

Proof. — Straightforward. \square

COROLLARY 3.9. — *We have $\Lambda_\Phi(\mathcal{T}_{\Phi, T_L}) \subseteq D((H_\Phi)_{\hat{\beta}}, \mu^o) \cap D({}_\alpha(H_\Phi), \mu)$ and $\Lambda_\Psi(\mathcal{T}_{\Psi, T_R}) \subseteq D({}_\alpha(H_\Psi), \mu) \cap D((H_\Psi)_\beta, \mu^o)$.*

Proof. — This is a corollary of the two previous propositions. \square

REMARK 3.10. — The invariance of operator-valued weights does not play a part in the previous propositions.

PROPOSITION 3.11. — *We have $(\omega_{v, \xi} \underset{\mu}{\beta} \star_\alpha id)(\Gamma(a)) \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$ for all elements $a \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$ and $v, \xi \in D(H_\beta, \mu^o)$.*

Proof. — By definition of the slice maps, we have:

$$\begin{aligned} (\omega_{v, \xi} \underset{\mu}{\beta} \star_\alpha id)(\Gamma(a))^* (\omega_{v, \xi} \underset{\mu}{\beta} \star_\alpha id)(\Gamma(a)) &= (\lambda_v^{\beta, \alpha})^* \Gamma(a^*) \lambda_\xi^{\beta, \alpha} (\lambda_\xi^{\beta, \alpha})^* \Gamma(a) \lambda_v^{\beta, \alpha} \\ &\leq \|\lambda_\xi^{\beta, \alpha}\|^2 (\omega_{v, v} \underset{\mu}{\beta} \star_\alpha id)(\Gamma(a^* a)) \\ &\leq \|R^{\beta, \mu^o}(\xi)\|^2 (\omega_{v, v} \underset{\mu}{\beta} \star_\alpha id)(\Gamma(a^* a)) \end{aligned}$$

Then, on one hand, we get, thanks to left invariance of T_L :

$$\begin{aligned} &T_L((\omega_{v, \xi} \underset{\mu}{\beta} \star_\alpha id)(\Gamma(a))^* (\omega_{v, \xi} \underset{\mu}{\beta} \star_\alpha id)(\Gamma(a))) \\ &\leq \|R^{\beta, \mu^o}(\xi)\|^2 T_L((\omega_{v, v} \underset{\mu}{\beta} \star_\alpha id)(\Gamma(a^* a))) \\ &= \|R^{\beta, \mu^o}(\xi)\|^2 (\omega_{v, v} \underset{\mu}{\beta} \star_\alpha id)(id \underset{\mu}{\beta} \star_\alpha T_L)(\Gamma(a^* a)) \\ &\leq \|R^{\beta, \mu^o}(\xi)\|^2 (\lambda_v^{\beta, \alpha})^* (T_L(a^* a) \underset{\mu}{\beta} \otimes_\alpha 1) \lambda_v^{\beta, \alpha} \\ &\leq \|R^{\beta, \mu^o}(\xi)\|^2 \|T_L(a^* a)\| \|\alpha(< v, v >_{\beta, \mu^o})\| 1 \\ &\leq \|R^{\beta, \mu^o}(\xi)\|^2 \|T_L(a^* a)\| \|R^{\beta, \mu^o}(v)\|^2 1 \end{aligned}$$

So, we get that $(\omega_{v, \xi} \underset{\mu}{\beta} \star_\alpha id)(\Gamma(a)) \in \mathcal{N}_{T_L}$. On the other hand, thanks to left invariance of T_L , we know that:

$$\Phi((\omega_{v, \xi} \underset{\mu}{\beta} \star_\alpha id)(\Gamma(a)))^* (\omega_{v, \xi} \underset{\mu}{\beta} \star_\alpha id)(\Gamma(a))$$

is less or equal to:

$$\begin{aligned} &\|R^{\beta, \mu^o}(\xi)\|^2 \Phi((\omega_{v, v} \underset{\mu}{\beta} \star_\alpha id)(\Gamma(a^* a))) \\ &= \|R^{\beta, \mu^o}(\xi)\|^2 \omega_{v, v}((id \underset{\mu}{\beta} \star_\alpha \Phi)(\Gamma(a^* a))) \\ &= \|R^{\beta, \mu^o}(\xi)\|^2 (T_L(a^* a)v|v) \leq \|R^{\beta, \mu^o}(\xi)\|^2 \|T_L(a^* a)\| \|v\|^2 < +\infty \end{aligned}$$

So, we get that $(\omega_{v,\xi} \underset{\mu}{\beta} \star_{\alpha} id)(\Gamma(a)) \in \mathcal{N}_{\Phi}$. \square

PROPOSITION 3.12. — *For all $v, w \in H$ and $a, b \in \mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L}$, we have:*

$$(v \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} \Lambda_{\Phi}(a) | w \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} \Lambda_{\Phi}(b)) = (T_L(b^*a)v | w)$$

For all $v, w \in H$ and $c, d \in \mathcal{N}_{\Psi} \cap \mathcal{N}_{T_R}$, we have:

$$(\Lambda_{\Psi}(c) \underset{\mu^o}{\hat{\alpha}} \otimes_{\beta} v | \Lambda_{\Psi}(d) \underset{\mu^o}{\hat{\alpha}} \otimes_{\beta} w) = (T_R(d^*c)v | w)$$

Proof. — Using 3.6 and 2.3, we get that:

$$\begin{aligned} (v \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} \Lambda_{\Phi}(a) | w \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} \Lambda_{\Phi}(b)) &= (\alpha(\langle \Lambda_{\Phi}(a), \Lambda_{\Phi}(b) \rangle_{\hat{\beta}, \mu^o})v | w) \\ &= (\alpha(\Lambda_{T_L}(b)^* \Lambda_{T_L}(a))v | w) \\ &= (\alpha(\pi_{\mu}(\alpha^{-1}(T_L(b^*a))))v | w) \end{aligned}$$

which gives the result after the identification of $\pi_{\mu}(N)$ with N . The second point is very similar. \square

LEMMA 3.13. — *Let $a \in \mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L}$ and $v \in D(H_{\beta}, \mu^o)$. The following sum:*

$$\sum_{i \in I} \xi_i \underset{\mu}{\beta} \otimes_{\alpha} \Lambda_{\Phi}((\omega_{v,\xi_i} \underset{\mu}{\beta} \star_{\alpha} id)(\Gamma(a)))$$

converges in $H \underset{\mu}{\beta} \otimes_{\alpha} H_{\Phi}$ for all (N^o, μ^o) -basis $(\xi_i)_{i \in I}$ of H_{β} and it does not depend on the (N^o, μ^o) -basis of H_{β} .

Proof. — By 3.11, we have $(\omega_{v,\xi_i} \underset{\mu}{\beta} \star_{\alpha} id)(\Gamma(a)) \in \mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L}$ for all $i \in I$, and the vectors $\xi_i \underset{\mu}{\beta} \otimes_{\alpha} \Lambda_{\Phi}((\omega_{v,\xi_i} \underset{\mu}{\beta} \star_{\alpha} id)(\Gamma(a)))$ are two-by-two orthogonal.

Normality and left invariance of Φ imply:

$$\begin{aligned} &\sum_{i \in I} \|\xi_i \underset{\mu}{\beta} \otimes_{\alpha} \Lambda_{\Phi}((\omega_{v,\xi_i} \underset{\mu}{\beta} \star_{\alpha} id)(\Gamma(a)))\|^2 \\ &= \sum_{i \in I} (\alpha(\langle \xi_i, \xi_i \rangle_{\beta, \mu^o}) \Lambda_{\Phi}((\omega_{v,\xi_i} \underset{\mu}{\beta} \star_{\alpha} id)(\Gamma(a))) | \Lambda_{\Phi}((\omega_{v,\xi_i} \underset{\mu}{\beta} \star_{\alpha} id)(\Gamma(a)))) \\ &= \Phi((\lambda_v^{\beta, \alpha})^* \Gamma(a^*) [\sum_{i \in I} \lambda_{\xi_i}^{\beta, \alpha} (\lambda_{\xi_i}^{\beta, \alpha})^* \lambda_{\xi_i}^{\beta, \alpha} (\lambda_{\xi_i}^{\beta, \alpha})^*] \Gamma(a) \lambda_v^{\beta, \alpha}) \\ &= \Phi((\omega_{v,v} \underset{\mu}{\beta} \star_{\alpha} id)(\Gamma(a^*a))) = ((id \underset{\mu}{\beta} \star_{\alpha} \Phi)(\Gamma(a^*a))v | v) = (T_L(a^*a)v | v) < \infty \end{aligned}$$

We deduce that the sum $\sum_{i \in I} \xi_i \underset{\mu}{\beta} \otimes_{\alpha} \Lambda_{\Phi}((\omega_{v,\xi_i} \underset{\mu}{\beta} \star_{\alpha} id)(\Gamma(a)))$ converges in $H \underset{\mu}{\beta} \otimes_{\alpha} H_{\Phi}$. To prove that the sum does not depend on the (N^o, μ^o) -basis, we

compute for all $b \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$ and $w \in D(H_\beta, \mu^o)$:

$$\begin{aligned}
 & \left(\sum_{i \in I} \xi_i \underset{\mu}{\beta \otimes_\alpha} \Lambda_\Phi((\omega_{v, \xi_i} \underset{\mu}{\beta \star_\alpha} id)(\Gamma(a))) \right) | w \underset{\mu}{\beta \otimes_\alpha} \Lambda_\Phi(b) \\
 &= \sum_{i \in I} (\alpha(\langle \xi_i, w \rangle_{\beta, \mu^o}) \Lambda_\Phi((\omega_{v, \xi_i} \underset{\mu}{\beta \star_\alpha} id)(\Gamma(a))) | \Lambda_\Phi(b)) \\
 &= \sum_{i \in I} \Phi(b^* \alpha(\langle \xi_i, w \rangle_{\beta, \mu^o}) (\omega_{v, \xi_i} \underset{\mu}{\beta \star_\alpha} id)(\Gamma(a))) \\
 &= \Phi(b^* \lambda_w^{\beta, \alpha} [\sum_{i \in I} \lambda_{\xi_i}^{\beta, \alpha} (\lambda_{\xi_i}^{\beta, \alpha})^*] \Gamma(a) \lambda_v^{\beta, \alpha}) = \Phi(b^* (\omega_{v, w} \underset{\mu}{\beta \star_\alpha} id)(\Gamma(a))).
 \end{aligned}$$

As $D(H_\beta, \mu^o) \odot \Lambda_\Phi(\mathcal{N}_{T_L} \cap \mathcal{N}_\Phi)$ is dense in $H \underset{\mu}{\beta \otimes_\alpha} H_\Phi$ and the last expression is independent of the (N^o, μ^o) -basis, we can conclude. \square

THEOREM 3.14. — *Let H be a Hilbert space on which M acts. There exists a unique isometry U_H , called **fundamental isometry**, from $H \underset{\mu^o}{\alpha \otimes_{\hat{\beta}}} H_\Phi$ to $H \underset{\mu}{\beta \otimes_\alpha} H_\Phi$ such that, for all (N^o, μ^o) -basis $(\xi_i)_{i \in I}$ of H_β , $a \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$ and $v \in D(H_\beta, \mu^o)$:*

$$U_H(v \underset{\mu^o}{\alpha \otimes_{\hat{\beta}}} \Lambda_\Phi(a)) = \sum_{i \in I} \xi_i \underset{\mu}{\beta \otimes_\alpha} \Lambda_\Phi((\omega_{v, \xi_i} \underset{\mu}{\beta \star_\alpha} id)(\Gamma(a)))$$

Proof. — By 3.13, we can define the following application:

$$\begin{aligned}
 \tilde{U} : D(H_\beta, \mu^o) \times \Lambda_\Phi(\mathcal{N}_T \cap \mathcal{N}_\Phi) &\rightarrow H \underset{\mu}{\beta \otimes_\alpha} H_\Phi \\
 (v, \Lambda_\Phi(a)) &\mapsto \sum_{i \in I} \xi_i \underset{\mu}{\beta \otimes_\alpha} \Lambda_\Phi((\omega_{v, \xi_i} \underset{\mu}{\beta \star_\alpha} id)(\Gamma(a)))
 \end{aligned}$$

Let $b \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$ and $w \in D(H_\beta, \mu^o)$. Then, by normality and left invariance of Φ , we have:

$$\begin{aligned}
 & (\tilde{U}(v, \Lambda_\Phi(a)) | \tilde{U}(w, \Lambda_\Phi(b))) \\
 &= \sum_{i, j \in I} (\alpha(\langle \xi_i, \xi_j \rangle_{\beta, \mu^o}) \Lambda_\Phi((\omega_{v, \xi_i} \underset{\mu}{\beta \star_\alpha} id)(\Gamma(a))) | \Lambda_\Phi((\omega_{w, \xi_j} \underset{\mu}{\beta \star_\alpha} id)(\Gamma(b)))) \\
 &= \sum_{i \in I} (\Lambda_\Phi(\alpha(\langle \xi_i, \xi_i \rangle_{\beta, \mu^o}) (\omega_{v, \xi_i} \underset{\mu}{\beta \star_\alpha} id)(\Gamma(a))) | \Lambda_\Phi((\omega_{w, \xi_i} \underset{\mu}{\beta \star_\alpha} id)(\Gamma(b)))) \\
 &= \sum_{i \in I} \Phi((\lambda_w^{\beta, \alpha})^* \Gamma(b^*) \lambda_{\xi_i}^{\beta, \alpha} \alpha(\langle \xi_i, \xi_i \rangle_{\beta, \mu^o}) (\lambda_{\xi_i}^{\beta, \alpha})^* \Gamma(a) \lambda_v^{\beta, \alpha}) \\
 &= \Phi((\lambda_w^{\beta, \alpha})^* \Gamma(b^*) [\sum_{i \in I} \lambda_{\xi_i}^{\beta, \alpha} (\lambda_{\xi_i}^{\beta, \alpha})^*] \Gamma(a) \lambda_v^{\beta, \alpha})
 \end{aligned}$$

Then, properties of (N^o, μ^o) -basis $(\xi_i)_{i \in I}$ of H_β imply that:

$$\begin{aligned} \Phi((\omega_{v,w} \underset{\mu}{\beta} \star_\alpha id)(\Gamma(b^*a))) &= \omega_{v,w}((id \underset{\mu}{\beta} \star_\alpha \Phi)(\Gamma(b^*a))) \\ &= \omega_{v,w}(T_L(b^*a)) = (T_L(b^*a)v|w) \end{aligned}$$

By 3.12, we get:

$$(\tilde{U}((v, \Lambda_\Phi(a))|\tilde{U}((w, \Lambda_\Phi(b)))) = (v \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} \Lambda_\Phi(a)|w \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} \Lambda_\Phi(b))$$

so that, from \tilde{U} , we can easily define a suitable application U_H which is independent of the (N^o, μ^o) -basis by 3.13. \square

One can define a right version of U_H from the right invariant weight:

THEOREM 3.15. — *Let H be a Hilbert space on which M acts. There exists a unique isometry U'_H from $H_{\Psi \underset{\mu^o}{\hat{\alpha}} \otimes_{\beta} H}$ to $H_{\Psi \underset{\mu}{\beta} \otimes_{\alpha} H}$ such that, for all (N, μ) -basis $(\eta_i)_{i \in I}$ of ${}_{\alpha}H$, $a \in \mathcal{N}_{T_R} \cap \mathcal{N}_{\Psi}$ and $v \in D({}_{\alpha}H, \mu)$:*

$$U'_H(\Lambda_{\Psi}(a) \underset{\mu^o}{\hat{\alpha}} \otimes_{\beta} v) = \sum_{i \in I} \Lambda_{\Psi}((id \underset{\mu}{\beta} \star_{\alpha} \omega_{v, \eta_i})(\Gamma(a))) \underset{\mu}{\beta} \otimes_{\alpha} \eta_i$$

3.3. Relations between the fundamental isometry and the co-product. —

PROPOSITION 3.16. — *We have $(1 \underset{N}{\beta} \otimes_{\alpha} J_{\Phi} e J_{\Phi}) U_H \rho_{\Lambda_{\Phi}(x)}^{\alpha, \hat{\beta}} = \Gamma(x) \rho_{J_{\Phi} \Lambda_{\Phi}(e)}^{\beta, \alpha}$ for all $e, x \in \mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L}$ and $(J_{\Psi} f J_{\Psi} \underset{N}{\beta} \otimes_{\alpha} 1) U'_H \lambda_{\Lambda_{\Psi}(y)}^{\hat{\alpha}, \beta} = \Gamma(y) \lambda_{J_{\Psi} \Lambda_{\Psi}(f)}^{\beta, \alpha}$ for all $f, y \in \mathcal{N}_{\Psi} \cap \mathcal{N}_{T_R}$.*

Proof. — Let $v \in D(H_\beta, \mu^o)$ and $(\xi_i)_{i \in I}$ a (N^o, μ^o) -basis of H_β . We have:

$$\begin{aligned} & (1 \underset{N}{\beta} \otimes_{\alpha} J_{\Phi} e J_{\Phi}) U_H (v \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} \Lambda_{\Phi}(x)) \\ &= \sum_{i \in I} \xi_i \underset{\mu}{\beta} \otimes_{\alpha} J_{\Phi} e J_{\Phi} \Lambda_{\Phi}((\omega_{v, \xi_i} \underset{\mu}{\beta} \star_{\alpha} id)(\Gamma(x))) \\ &= \sum_{i \in I} \xi_i \underset{\mu}{\beta} \otimes_{\alpha} (\omega_{v, \xi_i} \underset{\mu}{\beta} \star_{\alpha} id)(\Gamma(x)) J_{\Phi} \Lambda_{\Phi}(e) = \Gamma(x) (v \underset{\mu}{\beta} \otimes_{\alpha} J_{\Phi} \Lambda_{\Phi}(e)) \end{aligned}$$

By 3.6 and 3.8, we have $\Lambda_{\Phi}(x) \in D((H_{\Phi})_{\hat{\beta}}, \mu^o)$ and $J_{\Phi} \Lambda_{\Phi}(e) \in D({}_{\alpha}(H_{\Phi}), \mu)$ so that each term of the previous equality is continuous in v . Density of $D(H_\beta, \mu^o)$ in H finishes the proof. The last part is very similar. \square

PROPOSITION 3.17. — *For all $v, w \in D(H_\beta, \mu^o)$ and $a \in \mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L}$, we have:*

$$(\lambda_w^{\beta, \alpha})^* U_H (v \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} \Lambda_{\Phi}(a)) = \Lambda_{\Phi}((\omega_{v,w} \underset{\mu}{\beta} \star_{\alpha} id)(\Gamma(a)))$$

Also, for all $v', w' \in D({}_\alpha H, \mu)$ and $b \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$, we have:

$$(\rho_{w'}^{\beta, \alpha})^* U'_H(\Lambda_\Psi(b) \underset{\mu^o}{\alpha \otimes_{\hat{\beta}}} v') = \Lambda_\Psi((id \underset{\mu}{\beta \star_\alpha} \omega_{v', w'})(\Gamma(b)))$$

Proof. — Let $e \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$. By 3.16, we can compute:

$$\begin{aligned} J_\Phi e J_\Phi (\lambda_w^{\beta, \alpha})^* U_H(v \underset{\mu^o}{\alpha \otimes_{\hat{\beta}}} \Lambda_\Phi(a)) &= (\lambda_w^{\beta, \alpha})^* (1 \underset{N}{\beta \otimes_\alpha} J_\Phi e J_\Phi) U_H \rho_{\Lambda_\Phi(a)}^{\alpha, \hat{\beta}} v \\ &= (\lambda_w^{\beta, \alpha})^* \Gamma(a) \rho_{J_\Phi \Lambda_\Phi(e)}^{\beta, \alpha} v \\ &= (\omega_{v, w} \underset{\mu}{\beta \star_\alpha} id)(\Gamma(a)) J_\Phi \Lambda_\Phi(e) \\ &= J_\Phi e J_\Phi \Lambda_\Phi((\omega_{v, w} \underset{\mu}{\beta \star_\alpha} id)(\Gamma(a))) \end{aligned}$$

Density of $\mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$ in N finishes the proof. The second part is very similar. \square

COROLLARY 3.18. — For all $a \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$, $v \in D({}_\alpha H, \mu) \cap D(H_\beta, \mu^o)$ and $w \in D(H_\beta, \mu^o)$, we have:

$$(\omega_{v, w} * id)(U_H) \Lambda_\Phi(a) = \Lambda_\Phi((\omega_{v, w} \underset{\mu}{\beta \star_\alpha} id)(\Gamma(a)))$$

where we denote by $(\omega_{v, w} * id)(U_H)$ the operator $(\lambda_w^{\beta, \alpha})^* U_H \lambda_v^{\alpha, \hat{\beta}}$ of $\mathcal{L}(H_\Phi)$.

Proof. — Straightforward. \square

COROLLARY 3.19. — For all $e, x \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$ and $\eta \in D({}_\alpha H_\Phi, \mu^o)$, we have:

$$(id \underset{\mu}{\beta \star_\alpha} \omega_{J_\Phi \Lambda_\Phi(e), \eta})(\Gamma(x)) = (id * \omega_{\Lambda_\Phi(x), J_\Phi e * J_\Phi \eta})(U_H)$$

Also, for all $f, y \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$ and $\xi \in D((H_\Psi)_\beta, \mu^o)$, we have:

$$(\omega_{J_\Psi \Lambda_\Psi(f), \xi} \underset{\mu}{\beta \star_\alpha} id)(\Gamma(y)) = (\omega_{\Lambda_\Psi(y), J_\Psi f * J_\Psi \xi} * id)(U'_H)$$

Proof. — Straightforward by 3.16. \square

COROLLARY 3.20. — For all $a, b \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R} \cap \mathcal{N}_\Psi^* \cap \mathcal{N}_{T_R}^*$, we have:

$$(\omega_{\Lambda_\Psi(a), J_\Psi \Lambda_\Psi(b)} * id)(U'_H)^* = (\omega_{\Lambda_\Psi(a^*), J_\Psi \Lambda_\Psi(b^*)} * id)(U'_H)$$

Proof. — By 3.19, we have for all $e \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$:

$$\begin{aligned} (\omega_{\Lambda_\Psi(a), J_\Psi \Lambda_\Psi(e^* b)} * id)(U'_H)^* &= (\omega_{J_\Psi \Lambda_\Psi(e), J_\Psi \Lambda_\Psi(b)} \underset{\mu}{\beta \star_\alpha} id)(\Gamma(a))^* \\ &= (\omega_{J_\Psi \Lambda_\Psi(b), J_\Psi \Lambda_\Psi(e)} \underset{\mu}{\beta \star_\alpha} id)(\Gamma(a^*)) \\ &= (\omega_{\Lambda_\Psi(a^*), J_\Psi \Lambda_\Psi(b^* e)} * id)(U'_H). \end{aligned}$$

Let $(u_k)_{k \in K}$ be a family in $\mathcal{N}_\Psi \cap \mathcal{N}_\Psi^*$ such that $u_k \rightarrow 1$ in the $*$ -strong topology. We denote:

$$e_k = \frac{1}{\sqrt{\pi}} \int e^{-t^2} \sigma_t^\Psi(u_k) dt$$

For all $k \in K$, e_k and $\sigma_{-i/2}^\Psi(e_k^*)$ are bounded and belong to \mathcal{N}_Ψ and converge to 1 in the $*$ -strong topology so that $J_\Psi \Lambda_\Psi(b^* e_k) = \sigma_{-i/2}^\Psi(e_k^*) J_\Psi \Lambda_\Psi(b^*)$ converge to $J_\Psi \Lambda_\Psi(b^*)$ in norm of H_Ψ . Let $\xi, \eta \in D({}_\alpha H, \mu)$ and we compute:

$$\begin{aligned} ((\omega_{\Lambda_\Psi(a), J_\Psi \Lambda_\Psi(b)} * id)(U'_H)^* \xi | \eta) &= (J_\Psi \Lambda_\Psi(b) \underset{\mu}{\beta \otimes_\alpha} \xi | U'_H(\Lambda_\Psi(a) \underset{\mu^\circ}{\hat{\alpha} \otimes_\beta} \eta)) \\ &= \lim_{k \in K} (J_\Psi \Lambda_\Psi(e_k^* b) \underset{\mu}{\beta \otimes_\alpha} \xi | U'_H(\Lambda_\Psi(a) \underset{\mu^\circ}{\hat{\alpha} \otimes_\beta} \eta)) \\ &= \lim_{k \in K} ((\omega_{\Lambda_\Psi(a), J_\Psi \Lambda_\Psi(e_k^* b)} * id)(U'_H)^* \xi | \eta) \end{aligned}$$

By the previous computation, this last expression is equal to:

$$\begin{aligned} &\lim_{k \in K} ((\omega_{\Lambda_\Psi(a^*), J_\Psi \Lambda_\Psi(b^* e_k)} * id)(U'_H)^* \xi | \eta) \\ &= \lim_{k \in K} (U'_H(\Lambda_\Psi(a) \underset{\mu^\circ}{\hat{\alpha} \otimes_\beta} \xi) | J_\Psi \Lambda_\Psi(b^* e_k) \underset{\mu}{\beta \otimes_\alpha} \eta) \\ &= (U'_H(\Lambda_\Psi(a^*) \underset{\mu^\circ}{\hat{\alpha} \otimes_\beta} \xi) | J_\Psi \Lambda_\Psi(b^*) \underset{\mu}{\beta \otimes_\alpha} \eta) = ((\omega_{\Lambda_\Psi(a^*), J_\Psi \Lambda_\Psi(b^*)} * id)(U'_H)^* \xi | \eta) \end{aligned}$$

By density of $D({}_\alpha H, \mu)$ in H , the result holds. \square

3.4. Commutation relations. — In this section, we verify commutation relations which are necessary for U_H to be a pseudo-multiplicative unitary and we establish a link between U_H and Γ . We also have similar formulas for U'_H .

LEMMA 3.21. — *Let $\xi \in D(H_\beta, \mu^\circ)$ and $\eta \in D({}_\alpha H, \mu)$.*

- i) *For all $a \in \alpha(N)'$, we have $\lambda_\xi^{\beta, \alpha} \circ a = (1 \underset{N}{\beta \otimes_\alpha} a) \lambda_\xi^{\beta, \alpha}$.*
- ii) *For all $b \in \beta(N)'$, we have $\lambda_{b\xi}^{\beta, \alpha} = (b \underset{N}{\beta \otimes_\alpha} 1) \lambda_\xi^{\beta, \alpha}$.*
- iii) *For all $x \in \mathcal{D}(\sigma_{-i/2}^\mu)$, we have $\lambda_{\beta(x)\xi}^{\beta, \alpha} = \lambda_\xi^{\beta, \alpha} \circ \alpha(\sigma_{-i/2}^\mu(x))$.*
- iv) *For all $x \in \mathcal{D}(\sigma_{i/2}^\mu)$, we have $\rho_{\alpha(x)\eta}^{\beta, \alpha} = \rho_\eta^{\beta, \alpha} \circ \beta(\sigma_{i/2}^\mu(x))$.*

Proof. — Straightforward. \square

We recall that $\alpha(N)$ and $\beta(N)$ commute with $\hat{\beta}(N)'$.

PROPOSITION 3.22. — *For all $n \in N$, we have:*

- i) $U_H(1 \underset{N^\circ}{\alpha \otimes_{\hat{\beta}}} \alpha(n)) = (\alpha(n) \underset{N}{\beta \otimes_\alpha} 1) U_H;$
- ii) $U_H(1 \underset{N^\circ}{\alpha \otimes_{\hat{\beta}}} \beta(n)) = (1 \underset{N}{\beta \otimes_\alpha} \beta(n)) U_H;$
- iii) $U_H(\beta(n) \underset{N^\circ}{\alpha \otimes_{\hat{\beta}}} 1) = (1 \underset{N}{\beta \otimes_\alpha} \hat{\beta}(n)) U_H.$

Proof. — By 3.16, we can compute for all $n \in N$ and $e, x \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$:

$$\begin{aligned}
 (\alpha(n) \underset{N}{\beta} \otimes_\alpha J_\Phi e J_\Phi) U_H \rho_{\Lambda_\Phi(x)}^{\alpha, \hat{\beta}} &= (\alpha(n) \underset{N}{\beta} \otimes_\alpha 1) \Gamma(x) \rho_{J_\Phi \Lambda_\Phi(e)}^{\beta, \alpha} \\
 &= \Gamma(\alpha(n)x) \rho_{J_\Phi \Lambda_\Phi(e)}^{\beta, \alpha} \\
 &= (1 \underset{N}{\beta} \otimes_\alpha J_\Phi e J_\Phi) U_H \rho_{\Lambda_\Phi(\alpha(n)x)}^{\alpha, \hat{\beta}} \\
 &= (1 \underset{N}{\beta} \otimes_\alpha J_\Phi e J_\Phi) U_H (1 \underset{N^\circ}{\alpha} \otimes_{\hat{\beta}} \alpha(n)) \rho_{\Lambda_\Phi(x)}^{\alpha, \hat{\beta}}
 \end{aligned}$$

Usual arguments of density imply the first equality. The second one can be proved in a very similar way. By 3.16 and 3.21, we can compute for all $n \in \mathcal{T}_\mu$ and $e, x \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$:

$$\begin{aligned}
 (1 \underset{N}{\beta} \otimes_\alpha J_\Phi e J_\Phi \hat{\beta}(n)) U_H \rho_{\Lambda_\Phi(x)}^{\alpha, \hat{\beta}} &= \Gamma(x) \rho_{J_\Phi \Lambda_\Phi(e\alpha(n^*))}^{\beta, \alpha} \\
 &= \Gamma(x) \rho_{\alpha(\sigma_{-i/2}^\mu(n)) J_\Phi \Lambda_\Phi(e)}^{\beta, \alpha} \\
 &= \Gamma(x) \rho_{J_\Phi \Lambda_\Phi(e)}^{\beta, \alpha} \beta(n) \\
 &= (1 \underset{N}{\beta} \otimes_\alpha J_\Phi e J_\Phi) U_H \rho_{\Lambda_\Phi(x)}^{\alpha, \hat{\beta}} \beta(n) \\
 &= (1 \underset{N}{\beta} \otimes_\alpha J_\Phi e J_\Phi) U_H (\beta(n) \underset{N^\circ}{\alpha} \otimes_{\hat{\beta}} 1) \rho_{\Lambda_\Phi(x)}^{\alpha, \hat{\beta}}
 \end{aligned}$$

Density of \mathcal{T}_μ in N and normality of β and $\hat{\beta}$ finish the proof. \square

PROPOSITION 3.23. — *For all $x \in M' \cap \mathcal{L}(H)$, we have:*

$$U_H(x \underset{N^\circ}{\alpha} \otimes_{\hat{\beta}} 1) = (x \underset{N}{\beta} \otimes_\alpha 1) U_H$$

Proof. — For all $e, y \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$ and $x \in M' \cap \mathcal{L}(H) \subseteq \alpha(N)' \cap \beta(N)' \cap \mathcal{L}(H)$, we have by 3.16:

$$\begin{aligned}
 (x \underset{N}{\beta} \otimes_\alpha J_\Phi e J_\Phi) U_H \rho_{\Lambda_\Phi(y)}^{\alpha, \hat{\beta}} &= (x \underset{N}{\beta} \otimes_\alpha 1) \Gamma(y) \rho_{J_\Phi \Lambda_\Phi(e)}^{\beta, \alpha} \\
 &= \Gamma(y) \rho_{J_\Phi \Lambda_\Phi(e)}^{\beta, \alpha} x \\
 &= (1 \underset{N}{\beta} \otimes_\alpha J_\Phi e J_\Phi) U_H \rho_{\Lambda_\Phi(y)}^{\alpha, \hat{\beta}} x \\
 &= (1 \underset{N}{\beta} \otimes_\alpha J_\Phi e J_\Phi) U_H (x \underset{N^\circ}{\alpha} \otimes_{\hat{\beta}} 1) \rho_{\Lambda_\Phi(y)}^{\alpha, \hat{\beta}}
 \end{aligned}$$

Usual arguments of density imply the result. \square

COROLLARY 3.24. — *For all $n \in N$, we have:*

$$\begin{aligned} i) \quad U_{H_\Phi}(\hat{\beta}(n) \underset{N^o}{\alpha \otimes \hat{\beta}} 1) &= (\hat{\beta}(n) \underset{N}{\beta \otimes \alpha} 1) U_{H_\Phi} \\ ii) \quad U_{H_\Psi}(\hat{\alpha}(n) \underset{N^o}{\alpha \otimes \hat{\beta}} 1) &= (\hat{\alpha}(n) \underset{N}{\beta \otimes \alpha} 1) U_{H_\Psi} \end{aligned}$$

PROPOSITION 3.25. — We have $\Gamma(m)U_H = U_H(1 \underset{N^o}{\alpha \otimes \hat{\beta}} m)$ for all $m \in M$.

Proof. — By 3.16, we can compute for all $e, x \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$:

$$\begin{aligned} (1 \underset{N}{\beta \otimes \alpha} J_\Phi e J_\Phi) \Gamma(m) U_H \rho_{\Lambda_\Phi(x)}^{\alpha, \hat{\beta}} &= \Gamma(m) (1 \underset{N}{\beta \otimes \alpha} J_\Phi e J_\Phi) U_H \rho_{\Lambda_\Phi(x)}^{\alpha, \hat{\beta}} \\ &= \Gamma(mx) \rho_{J_\Phi \Lambda_\Phi(e)}^{\beta, \alpha} \\ &= (1 \underset{N}{\beta \otimes \alpha} J_\Phi e J_\Phi) U_H \rho_{\Lambda_\Phi(mx)}^{\alpha, \hat{\beta}} \\ &= (1 \underset{N}{\beta \otimes \alpha} J_\Phi e J_\Phi) U_H (1 \underset{N^o}{\alpha \otimes \hat{\beta}} m) \rho_{\Lambda_\Phi(x)}^{\alpha, \hat{\beta}} \end{aligned}$$

Usual arguments of density imply the result. \square

3.5. Unitarity of the fundamental isometry. — To prove unitary of U_H (resp. U'_H), we establish a reciprocity law where both left and right operator-valued weights are at stake.

3.5.1. *First technical result.* — We establish results needed for 3.5.3. In the following proposition, we compute some functions θ defined in section 2.2.

PROPOSITION 3.26. — We have for all $c \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$, $m \in (\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})^*$ and $v \in D(H_\beta, \mu^o)$:

$$\theta^{\beta, \mu^o}(v, J_\Psi \Lambda_\Psi(c))m = (\lambda_{\Lambda_\Psi(m^*)}^{\hat{\alpha}, \beta})^* \rho_v^{\hat{\alpha}, \beta} J_\Psi c^* J_\Psi$$

Proof. — Let $x \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$. On one hand, we get by 3.6 and 3.8:

$$\begin{aligned} \theta^{\beta, \mu^o}(v, J_\Psi \Lambda_\Psi(c))m \Lambda_\Psi(x) &= R^{\beta, \mu^o}(v) R^{\beta, \mu^o}(J_\Psi \Lambda_\Psi(c))^* \Lambda_\Psi(mx) \\ &= R^{\beta, \mu^o}(v) J_\mu \Lambda_{T_R}(c)^* J_\Psi \Lambda_\Psi(mx). \end{aligned}$$

On the other hand, if $c \in \mathcal{T}_{\Psi, T_R}$, then we have by 3.12:

$$\begin{aligned} (\lambda_{\Lambda_\Psi(m^*)}^{\hat{\alpha}, \beta})^* \rho_v^{\hat{\alpha}, \beta} J_\Psi c^* J_\Psi \Lambda_\Psi(x) &= (\lambda_{\Lambda_\Psi(m^*)}^{\hat{\alpha}, \beta})^* (J_\Psi c^* J_\Psi \Lambda_\Psi(x) \underset{\mu^o}{\hat{\alpha} \otimes \beta} v) \\ &= T_R(mx \sigma_{-i/2}^\Psi(c))v \\ &= R^{\beta, \mu^o}(v) J_\mu \Lambda_\mu(\beta^{-1}(T_R(\sigma_{i/2}^\Psi(c^*)x^*m^*))) \\ &= R^{\beta, \mu^o}(v) J_\mu \Lambda_{T_R}(c)^* J_\Psi \Lambda_\Psi(mx) \end{aligned}$$

We obtain:

$$(\lambda_{\Lambda_\Psi(m^*)}^{\hat{\alpha}, \beta})^* \rho_v^{\hat{\alpha}, \beta} J_\Psi c^* J_\Psi \Lambda_\Psi(x) = R^{\beta, \mu^o}(v) J_\mu \Lambda_{T_R}(c)^* J_\Psi \Lambda_\Psi(mx)$$

for all $c \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$ by normality which finishes the proof. \square

COROLLARY 3.27. — *Let $a \in (\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})^*(\mathcal{N}_\Phi \cap \mathcal{N}_{T_L})$. If $c \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$, $e \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$ and $\xi \in H_\Psi, \eta \in D(\alpha(H_\Phi), \mu)$, $u \in H$, $v \in D(H_\beta, \mu^o)$, then we have:*

$$\begin{aligned} & (v \underset{\mu}{\beta} \otimes_\alpha (\lambda_{J_\Psi \Lambda_\Psi(c)}^{\beta, \alpha})^* U_{H_\Psi}(\xi \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} \Lambda_\Phi(a)) | u \underset{\mu}{\beta} \otimes_\alpha J_\Phi e^* J_\Phi \eta) \\ &= (J_\Psi c^* J_\Psi \xi \underset{\mu^o}{\hat{\alpha}} \otimes_\beta v | \Lambda_\Psi((id \underset{\mu}{\beta} \star_\alpha \omega_{\eta, J_\Phi \Lambda_\Phi(e)}) (\Gamma(a^*))) \underset{\mu^o}{\hat{\alpha}} \otimes_\beta u) \end{aligned}$$

Proof. — By 3.16 and 3.26, we can compute:

$$\begin{aligned} & (v \underset{\mu}{\beta} \otimes_\alpha (\lambda_{J_\Psi \Lambda_\Psi(c)}^{\beta, \alpha})^* U_{H_\Psi}(\xi \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} \Lambda_\Phi(a)) | u \underset{\mu}{\beta} \otimes_\alpha J_\Phi e^* J_\Phi \eta) \\ &= ((\rho_\eta^{\beta, \alpha})^* \lambda_v^{\beta, \alpha} (\lambda_{J_\Psi \Lambda_\Psi(c)}^{\beta, \alpha})^* (1 \underset{N}{\beta} \otimes_\alpha J_\Phi e^* J_\Phi) U_{H_\Psi}(\xi \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} \Lambda_\Phi(a)) | u) \\ &= ((\rho_\eta^{\beta, \alpha})^* \lambda_v^{\beta, \alpha} (\lambda_{J_\Psi \Lambda_\Psi(c)}^{\beta, \alpha})^* \Gamma(a) \rho_{J_\Phi \Lambda_\Phi(e)}^{\beta, \alpha} \xi | u) \\ &= \theta^{\beta, \mu^o}(v, J_\Psi \Lambda_\Psi(c)) (\rho_\eta^{\beta, \alpha})^* \Gamma(a) \rho_{J_\Phi \Lambda_\Phi(e)}^{\beta, \alpha} \xi | u) \\ &= ((\lambda_{\Lambda_\Psi}^{\hat{\alpha}, \beta} ((id \underset{\mu}{\beta} \star_\alpha \omega_{\eta, J_\Phi \Lambda_\Phi(e)}) (\Gamma(a^*)))^* \rho_v^{\hat{\alpha}, \beta} J_\Psi c^* J_\Psi \xi | u) \\ &= (J_\Psi c^* J_\Psi \xi \underset{\mu^o}{\hat{\alpha}} \otimes_\beta v | \Lambda_\Psi((id \underset{\mu}{\beta} \star_\alpha \omega_{\eta, J_\Phi \Lambda_\Phi(e)}) (\Gamma(a^*))) \underset{\mu^o}{\hat{\alpha}} \otimes_\beta u) \end{aligned}$$

\square

3.5.2. *Second technical result.* — In this section, results only depend on 3.16 and co-product relation but not on the previous technical result. Let \mathcal{H} be an other Hilbert space on which M acts.

LEMMA 3.28. — *Let $a, e \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$, $\xi \in D(\mathcal{H}_\beta, \mu^o)$, $\eta \in D(\alpha H, \mu)$, and $\zeta \in \mathcal{H}$. We have:*

$$\begin{aligned} & (1 \underset{N}{\beta} \otimes_\alpha J_\Phi e J_\Phi) U_H(\eta \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} [(\lambda_\xi^{\beta, \alpha})^* U_{\mathcal{H}}(\zeta \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} \Lambda_\Phi(a))]) \\ &= (\lambda_\xi^{\beta, \alpha} \underset{N}{\beta} \otimes_\alpha 1)^* (id \underset{N}{\beta} \star_\alpha \Gamma)(\Gamma(a)) (\zeta \underset{\mu}{\beta} \otimes_\alpha \eta \underset{\mu}{\beta} \otimes_\alpha J_\Phi \Lambda_\Phi(e)) \end{aligned}$$

Proof. — First let assume $\zeta \in D(\mathcal{H}_\beta, \mu^o)$. By 3.17 and 3.16, we can compute:

$$\begin{aligned} & (1 \underset{N}{\beta} \otimes_\alpha J_\Phi e J_\Phi) U_H(\eta \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} [(\lambda_\xi^{\beta, \alpha})^* U_{\mathcal{H}}(\zeta \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} \Lambda_\Phi(a))]) \\ &= (1 \underset{N}{\beta} \otimes_\alpha J_\Phi e J_\Phi) U_H(\eta \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} \Lambda_\Phi((\omega_{\zeta, \xi} \underset{\mu}{\beta} \star_\alpha id)(\Gamma(a))) \\ &= \Gamma((\omega_{\zeta, \xi} \underset{\mu}{\beta} \star_\alpha id)(\Gamma(a))) (\eta \underset{\mu}{\beta} \otimes_\alpha J_\Phi \Lambda_\Phi(e)) \\ &= (\lambda_\xi^{\beta, \alpha} \underset{N}{\beta} \otimes_\alpha 1)^* (id \underset{N}{\beta} \star_\alpha \Gamma)(\Gamma(a)) (\zeta \underset{\mu}{\beta} \otimes_\alpha \eta \underset{\mu}{\beta} \otimes_\alpha J_\Phi \Lambda_\Phi(e)) \end{aligned}$$

So, we get the result for all $\zeta \in D(\mathcal{H}_\beta, \mu^o)$. The first term of the equality is continuous in ζ because $\eta \in D({}_\alpha H, \mu)$ and $\Lambda_\Phi(a) \in D((H_\Phi)_\beta, \mu^o)$. Also, since $\eta \in D({}_\alpha H, \mu)$ and $\Lambda_\Phi(a) \in D((H_\Phi)_\beta, \mu^o)$, the last term of the equality is continuous in ζ . Density of $D(\mathcal{H}_\beta, \mu^o)$ in \mathcal{H} finishes the proof. \square

LEMMA 3.29. — *The sum $\sum_{i \in I} \eta_i \underset{\mu^o}{\alpha \otimes \hat{\beta}} [(\lambda_\xi^{\beta, \alpha})^* U_{\mathcal{H}}((\rho_{\eta_i}^{\beta, \alpha})^* \Xi \underset{\mu^o}{\alpha \otimes \hat{\beta}} \Lambda_\Phi(a))]$ converges for all $\xi \in D(\mathcal{H}_\beta, \mu^o)$, $\Xi \in \mathcal{H} \underset{\mu}{\beta \otimes \alpha} H$, $a \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$ and (N, μ) -basis $(\eta_i)_{i \in I}$ of ${}_\alpha H$.*

Proof. — First, observe that $\eta_i \underset{\mu^o}{\alpha \otimes \hat{\beta}} [(\lambda_\xi^{\beta, \alpha})^* U_{\mathcal{H}}((\rho_{\eta_i}^{\beta, \alpha})^* \Xi \underset{\mu^o}{\alpha \otimes \hat{\beta}} \Lambda_\Phi(a))]$ are orthogonal. To compute, we put: $\Omega_i = \rho_{\eta_i}^{\beta, \alpha})^* \Xi \underset{\mu^o}{\alpha \otimes \hat{\beta}} \Lambda_\Phi(a)$. By 3.21 and 3.22, we have:

$$\begin{aligned} & \| \eta_i \underset{\mu^o}{\alpha \otimes \hat{\beta}} [(\lambda_\xi^{\beta, \alpha})^* U_{\mathcal{H}}(\Omega_i)] \|^2 \\ &= (\hat{\beta}(\langle \eta_i, \eta_i \rangle_{\alpha, \mu}) (\lambda_\xi^{\beta, \alpha})^* U_{\mathcal{H}}(\Omega_i) | (\lambda_\xi^{\beta, \alpha})^* U_{\mathcal{H}}(\Omega_i)) \\ &= ((\lambda_\xi^{\beta, \alpha})^* (1 \underset{\mu}{\beta \otimes \alpha} \hat{\beta}(\langle \eta_i, \eta_i \rangle_{\alpha, \mu})) U_{\mathcal{H}}(\Omega_i) | (\lambda_\xi^{\beta, \alpha})^* U_{\mathcal{H}}(\Omega_i)) \\ &= ((\lambda_\xi^{\beta, \alpha})^* U_{\mathcal{H}}(\beta(\langle \eta_i, \eta_i \rangle_{\alpha, \mu})(\Omega_i)) | (\lambda_\xi^{\beta, \alpha})^* U_{\mathcal{H}}(\Omega_i)) \\ &= (\lambda_\xi^{\beta, \alpha} (\lambda_\xi^{\beta, \alpha})^* U_{\mathcal{H}}(\Omega_i) | U_{\mathcal{H}}(\Omega_i)) \end{aligned}$$

By 3.12, it follows that we have, for all $i \in I$:

$$\begin{aligned} & \| \eta_i \underset{\mu^o}{\alpha \otimes \hat{\beta}} [(\lambda_\xi^{\beta, \alpha})^* U_{\mathcal{H}}((\rho_{\eta_i}^{\beta, \alpha})^* \Xi \underset{\mu^o}{\alpha \otimes \hat{\beta}} \Lambda_\Phi(a))] \|^2 \\ &\leq \| R^{\beta, \alpha}(\xi) \|^2 ((\rho_{\eta_i}^{\beta, \alpha})^* \Xi \underset{\mu^o}{\alpha \otimes \hat{\beta}} \Lambda_\Phi(a) | (\rho_{\eta_i}^{\beta, \alpha})^* \Xi \underset{\mu^o}{\alpha \otimes \hat{\beta}} \Lambda_\Phi(a)) \\ &\leq \| R^{\beta, \alpha}(\xi) \|^2 (T_L(a^* a) (\rho_{\eta_i}^{\beta, \alpha})^* \Xi | (\rho_{\eta_i}^{\beta, \alpha})^* \Xi) \\ &\leq \| R^{\beta, \alpha}(\xi) \|^2 \| T(a^* a) \| ((\rho_{\eta_i}^{\beta, \alpha})^* \Xi | (\rho_{\eta_i}^{\beta, \alpha})^* \Xi) \end{aligned}$$

So, we can sum over $i \in I$ to get that:

$$\sum_{i \in I} \| \eta_i \underset{\mu^o}{\alpha \otimes \hat{\beta}} [(\lambda_\xi^{\beta, \alpha})^* U_{\mathcal{H}}((\rho_{\eta_i}^{\beta, \alpha})^* \Xi \underset{\mu^o}{\alpha \otimes \hat{\beta}} \Lambda_\Phi(a))] \|^2$$

is less or equal to $\| R^{\beta, \alpha}(\xi) \|^2 \| T(a^* a) \| \| \Xi \|^2 < \infty$. That's why the sum converges. \square

PROPOSITION 3.30. — *Let $a, e \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$, $\Xi \in \mathcal{H}_{\beta \otimes_\alpha H}$, $\xi \in D(\mathcal{H}_\beta, \mu^o)$, $\eta \in D({}_\alpha(H_\Phi), \mu)$ and $(\eta_i)_{i \in I}$ a (N, μ) -basis of ${}_\alpha H$. We have:*

$$\begin{aligned} & (\rho_{J_\Phi e J_\Phi \eta}^{\beta, \alpha})^* U_H \left(\sum_{i \in I} \eta_i \underset{\mu^o}{\alpha \otimes_{\hat{\beta}}} [(\lambda_\xi^{\beta, \alpha})^* U_{\mathcal{H}}((\rho_{\eta_i}^{\beta, \alpha})^* \Xi \underset{\mu^o}{\alpha \otimes_{\hat{\beta}}} \Lambda_\Phi(a))] \right) \\ &= (\lambda_\xi^{\beta, \alpha})^* \Gamma((id \underset{\mu}{\beta \star_\alpha} \omega_{J_\Phi \Lambda_\Phi(e), \eta})(\Gamma(a))) \Xi \end{aligned}$$

Proof. — The existence of the first term comes from the previous lemma. By 3.28 and the co-product relation, we can compute:

$$\begin{aligned} & \sum_{i \in I} (\rho_\eta^{\beta, \alpha})^* (1 \underset{N}{\beta \otimes_\alpha} J_\Phi e J_\Phi) U_H (\eta_i \underset{\mu^o}{\alpha \otimes_{\hat{\beta}}} [(\lambda_\xi^{\beta, \alpha})^* U_{\mathcal{H}}((\rho_{\eta_i}^{\beta, \alpha})^* \Xi \underset{\mu^o}{\alpha \otimes_{\hat{\beta}}} \Lambda_\Phi(a))]) \\ &= \sum_{i \in I} (\rho_\eta^{\beta, \alpha})^* (\lambda_\xi^{\beta, \alpha} \underset{N}{\beta \otimes_\alpha} 1)^* (id \underset{N}{\beta \star_\alpha} \Gamma)(\Gamma(a)) ((\rho_{\eta_i}^{\beta, \alpha})^* \Xi \underset{\mu}{\beta \otimes_\alpha} \eta_i \underset{\mu}{\beta \otimes_\alpha} J_\Phi \Lambda_\Phi(e)) \\ &= (\rho_\eta^{\beta, \alpha})^* (\lambda_\xi^{\beta, \alpha} \underset{N}{\beta \otimes_\alpha} 1)^* (\Gamma \underset{N}{\beta \star_\alpha} id)(\Gamma(a)) ([\sum_{i \in I} \rho_{\eta_i}^{\beta, \alpha} (\rho_{\eta_i}^{\beta, \alpha})^*] \Xi \underset{\mu}{\beta \otimes_\alpha} J_\Phi \Lambda_\Phi(e)) \\ &= (\rho_\eta^{\beta, \alpha})^* (\lambda_\xi^{\beta, \alpha} \underset{N}{\beta \otimes_\alpha} 1)^* (\Gamma \underset{N}{\beta \star_\alpha} id)(\Gamma(a)) (\Xi \underset{\mu}{\beta \otimes_\alpha} J_\Phi \Lambda_\Phi(e)) \\ &= (\lambda_\xi^{\beta, \alpha})^* (1 \underset{N}{\beta \otimes_\alpha} \rho_\eta^{\beta, \alpha})^* (\Gamma \underset{N}{\beta \star_\alpha} id)(\Gamma(a)) (\Xi \underset{\mu}{\beta \otimes_\alpha} J_\Phi \Lambda_\Phi(e)) \\ &= (\lambda_\xi^{\beta, \alpha})^* \Gamma((id \underset{\mu}{\beta \star_\alpha} \omega_{J_\Phi \Lambda_\Phi(e), \eta})(\Gamma(a))) \Xi \end{aligned}$$

□

With results of the two last sections in hand, we can prove now a reciprocity law where \mathcal{H} will be equal to H_Ψ .

3.5.3. *Reciprocity law.* — For all monotone increasing net $(e_k)_{k \in K}$ in $\mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$ of limit equal to 1, the following $(\omega_{J_\Psi \Lambda_\Psi(e_k)})_{k \in K}$ is monotone increasing and converges to Ψ . So, for all $x \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$, $(\omega_{J_\Psi \Lambda_\Psi(e_k)} \underset{\mu}{\beta \star_\alpha} id)(\Gamma(x))$ converges to $(\Psi \underset{\mu}{\beta \star_\alpha} id)(\Gamma(x))$ in the weak topology. We denote $\zeta_k = J_\Psi \Lambda_\Psi(e_k^* e_k) \in D((H_\Psi)_\beta, \mu^o)$ for all $k \in K$.

PROPOSITION 3.31. — *For all $a \in (\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})^* (\mathcal{N}_\Phi \cap \mathcal{N}_{T_L})$, $e \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$, $b \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$, $c \in \mathcal{T}_{\Psi, T_R}$, $v \in D(H_\beta, \mu^o)$, $\eta \in D({}_\alpha(H_\Phi), \mu)$ and (N, μ) -basis of ${}_\alpha H$, $(\eta_i)_{i \in I}$, we have that the image of:*

$$\sum_{i \in I} \eta_i \underset{\mu^o}{\alpha \otimes_{\hat{\beta}}} [(\lambda_{\zeta_k}^{\beta, \alpha})^* U_{H_\Psi} ([(\rho_{\eta_i}^{\beta, \alpha})^* U'_H (J_\Psi c^* J_\Psi \Lambda_\Psi(b) \underset{\mu^o}{\hat{\alpha} \otimes_{\hat{\beta}}} v)] \underset{\mu^o}{\alpha \otimes_{\hat{\beta}}} \Lambda_\Phi(a))]$$

by $(\rho_{J_\Phi e^* J_\Phi \eta}^{\beta, \alpha})^* U_H$ converges, in the weak topology, to:

$$(\rho_{J_\Phi e^* J_\Phi \eta}^{\beta, \alpha})^* (v \underset{\mu}{\beta \otimes \alpha} (\lambda_{J_\Psi \Lambda_\Psi(c)}^{\beta, \alpha})^* U_{H_\Psi} (\Lambda_\Psi(b) \underset{\mu^o}{\alpha \otimes \beta} \Lambda_\Phi(a)))$$

Proof. — Let $u \in H$. We compute the value of the scalar product of:

$$U_H \left(\sum_{i \in I} \eta_i \underset{\mu^o}{\alpha \otimes \beta} [(\lambda_{\zeta_k}^{\beta, \alpha})^* U_{H_\Psi} ((\rho_{\eta_i}^{\beta, \alpha})^* U'_H (\Lambda_\Psi(bc) \underset{\mu^o}{\alpha \otimes \beta} v)) \underset{\mu^o}{\alpha \otimes \beta} \Lambda_\Phi(a)) \right]$$

by $u \underset{\mu}{\beta \otimes \alpha} J_\Phi e^* J_\Phi \eta$. By 3.30, we get that it is equal to:

$$(\Gamma((id \underset{\mu}{\beta \star \alpha} \omega_{J_\Phi \Lambda_\Phi(e), \eta})(\Gamma(a))) U'_H (\Lambda_\Psi(bc) \underset{\mu^o}{\alpha \otimes \beta} v) | \zeta_k \underset{\mu}{\beta \otimes \alpha} u)$$

By the right version of 3.25, this is equal to:

$$(U'_H (\Lambda_\Psi((id \underset{\mu}{\beta \star \alpha} \omega_{J_\Phi \Lambda_\Phi(e), \eta})(\Gamma(a)) bc) \underset{\mu^o}{\alpha \otimes \beta} v) | \zeta_k \underset{\mu}{\beta \otimes \alpha} u)$$

By 3.16, we obtain:

$$((\omega_{J_\Psi \Lambda_\Psi(e_k)} \underset{\mu}{\beta \star \alpha} id)(\Gamma((id \underset{\mu}{\beta \star \alpha} \omega_{J_\Phi \Lambda_\Phi(e), \eta})(\Gamma(a)) bc)) v | u)$$

which converges to:

$$((\Psi \underset{\mu}{\beta \star \alpha} id)(\Gamma((id \underset{\mu}{\beta \star \alpha} \omega_{J_\Phi \Lambda_\Phi(e), \eta})(\Gamma(a))) bc) v | u)$$

Now, by right invariance of T_R , 3.12 and 3.27, we can compute this last expression:

$$\begin{aligned} & ((\Psi \underset{\mu}{\beta \star \alpha} id)(\Gamma((id \underset{\mu}{\beta \star \alpha} \omega_{J_\Phi \Lambda_\Phi(e), \eta})(\Gamma(a))) bc) v | u) \\ &= (T_R((id \underset{\mu}{\beta \star \alpha} \omega_{J_\Phi \Lambda_\Phi(e), \eta})(\Gamma(a)) bc) v | u) \\ &= (\Lambda_\Psi(bc) \underset{\mu^o}{\alpha \otimes \beta} v | \Lambda_\Psi((id \underset{\mu}{\beta \star \alpha} \omega_{\eta, J_\Phi \Lambda_\Phi(e)})(\Gamma(a^*)))) \underset{\mu^o}{\alpha \otimes \beta} u) \\ &= (v \underset{\mu}{\beta \otimes \alpha} (\lambda_{\Lambda_\Psi(\sigma_{-i}^*(c^*))}^{\beta, \alpha})^* U_{H_\Psi} (\Lambda_\Psi(b) \underset{\mu^o}{\alpha \otimes \beta} \Lambda_\Phi(a)) | u \underset{\mu}{\beta \otimes \alpha} J_\Phi e^* J_\Phi \eta) \end{aligned}$$

which finishes the proof. \square

Let $(\eta_i)_{i \in I}$ be a (N, μ) -basis of ${}_\alpha H$. For all finite subset J of I , we denote by P_J the projection $\sum_{i \in J} \theta^{\alpha, \mu}(\eta_i, \eta_i) \in \alpha(N)'$ so that:

$$\sum_{i \in J} \rho_{\eta_i}^{\beta, \alpha} (\rho_{\eta_i}^{\beta, \alpha})^* = 1 \underset{N}{\beta \otimes \alpha} P_J$$

For all $e \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$, we also denote by P_J^e :

$$1 \underset{N}{\beta \otimes \alpha} J_\Phi e^* J_\Phi P_J J_\Phi e J_\Phi = \sum_{i \in J} \rho_{J_\Phi e^* J_\Phi \eta_i}^{\beta, \alpha} (\rho_{J_\Phi e^* J_\Phi \eta_i}^{\beta, \alpha})^*$$

COROLLARY 3.32. — *For all $a \in (\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})^*(\mathcal{N}_\Phi \cap \mathcal{N}_{T_L})$, $b \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$, and $c \in \mathcal{T}_{\Psi, T_R}$, $v \in D(H_\beta, \mu^o)$, $e \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$ and J finite subset of I , we have:*

$$P_J^e U_H \left(\sum_{i \in I} \eta_i \underset{\mu^o}{\alpha \otimes \hat{\beta}} [(\lambda_{\zeta_k}^{\beta, \alpha})^* U_{H_\Psi} ((\rho_{\eta_i}^{\beta, \alpha})^* U'_H (J_\Psi c^* J_\Psi \Lambda_\Psi(b) \underset{\mu^o}{\alpha \otimes \hat{\beta}} v))] \underset{\mu^o}{\alpha \otimes \hat{\beta}} \Lambda_\Phi(a) \right]$$

converges, in the weak topology, to:

$$P_J^e(v \underset{\mu}{\beta \otimes \alpha} (\lambda_{J_\Psi \Lambda_\Psi(c)}^{\beta, \alpha})^* U_{H_\Psi} (\Lambda_\Psi(b) \underset{\mu^o}{\alpha \otimes \hat{\beta}} \Lambda_\Phi(a)))$$

Proof. — We apply to the reciprocity law $\rho_{J_\Phi e^* J_\Phi \eta}^{\beta, \alpha}$ which is a continuous linear operator of H in $H \underset{\mu}{\beta \otimes \alpha} H_\Phi$, and also a continuous linear operator of H with weak topology in $H \underset{\mu}{\beta \otimes \alpha} H_\Phi$ with weak topology. Then, we take finite sums. \square

Until the end of the section, we denote by \mathcal{H}_Φ the closed linear span in H_Φ of $(\lambda_w^{\beta, \alpha})^* U_{H_\Psi}(v \underset{\mu^o}{\alpha \otimes \hat{\beta}} \Lambda_\Phi(a))$ where $v \in H_\Psi$, $w \in J_\Psi \Lambda_\Psi(\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})$, and $a \in (\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})^* \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$. By the third relation of lemma 3.21 (resp. proposition 3.22), α (resp. $\hat{\beta}$) is a non-degenerated (resp. anti-) representation of N on \mathcal{H}_Φ .

LEMMA 3.33. — *Let $a \in (\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})^*(\mathcal{N}_\Phi \cap \mathcal{N}_T)$, $b \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$, $c \in \mathcal{T}_{\Psi, T_R}$, $v \in D(H_\beta, \mu^o)$ and $(\eta_i)_{i \in I}$ a (N, μ) -basis of $\underset{\mu}{\alpha} H$. We put, for all $k \in K$:*

$$\Xi_k = \left(\sum_{i \in I} \eta_i \underset{\mu^o}{\alpha \otimes \hat{\beta}} [(\lambda_{\zeta_k}^{\beta, \alpha})^* U_{H_\Psi} ((\rho_{\eta_i}^{\beta, \alpha})^* U'_H (J_\Psi c^* J_\Psi \Lambda_\Psi(b) \underset{\mu^o}{\alpha \otimes \hat{\beta}} v))] \underset{\mu^o}{\alpha \otimes \hat{\beta}} \Lambda_\Phi(a) \right]$$

Then the net $(\Xi_k)_{k \in K}$ is bounded.

Proof. — Let $\Xi = v \underset{\mu}{\beta \otimes \alpha} (\lambda_{J_\Psi \Lambda_\Psi(c)}^{\beta, \alpha})^* U_{H_\Psi} (\Lambda_\Psi(b) \underset{\mu^o}{\alpha \otimes \hat{\beta}} \Lambda_\Phi(a))$. By the previous corollary, we know that $P_J^e U_H \Xi_k$ weakly converges to $P_J^e \Xi$, so that:

$$\lim_{J, ||e|| \leq 1} \lim_k P_J^e U_H \Xi_k = \Xi$$

Consequently, there exists $C \in \mathbb{R}^+$ such that:

$$\sup_{J, ||e|| \leq 1} \sup_k ||P_J^e U_H \Xi_k|| \leq C$$

and, the interversion of the supremum gives:

$$C \geq \sup_k \sup_{J, ||e|| \leq 1} ||P_J^e U_H \Xi_k|| = \sup_k ||U_H \Xi_k|| = \sup_k ||\Xi_k||$$

\square

COROLLARY 3.34. — For all $a \in (\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})^*(\mathcal{N}_\Phi \cap \mathcal{N}_T)$, $b \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$, $c \in \mathcal{T}_{\Psi, T_R}$, $v \in D(H_\beta, \mu^o)$ and $(\eta_i)_{i \in I}$ a (N, μ) -basis of ${}_\alpha H$, we put:

$$\Xi_k = \left(\sum_{i \in I} \eta_i {}_\alpha \otimes_{\beta} {}_{\mu^o} [(\lambda_{\zeta_k}^{\beta, \alpha})^* U_{H_\Psi}((\rho_{\eta_i}^{\beta, \alpha})^* U'_H(J_\Psi c^* J_\Psi \Lambda_\Psi(b) {}_{\hat{\alpha}} \otimes_{\beta} v)) {}_\alpha \otimes_{\hat{\beta}} \Lambda_\Phi(a)] \right)$$

for all $k \in K$, and:

$$\Xi = v {}_\beta \otimes_{\alpha} {}_{\mu} (\lambda_{J_\Psi \Lambda_\Psi(c)}^{\beta, \alpha})^* U_{H_\Psi}(\Lambda_\Psi(b) {}_\alpha \otimes_{\hat{\beta}} \Lambda_\Phi(a))$$

Then $U_H \Xi_k$ converges to Ξ in the weak topology.

Proof. — Let $\Theta \in H {}_\beta \otimes_{\alpha} {}_{\mu} H_\Phi$ and $\epsilon > 0$. Then, there exists $e \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$ of norm less than equal to 1 and a finite subset J of I such that $\|(1 - P_J^e)\Theta\| \leq \epsilon$. By 3.32, there also exists k_0 such that $|(P_J^e U_H \Xi_k - P_J^e \Xi)| \leq \epsilon$ for all $k \geq k_0$. Then, we get:

$$\begin{aligned} & |(U_H \Xi_k - \Xi)| \\ & \leq |(U_H \Xi_k - P_J^e U_H \Xi_k)| + |(P_J^e U_H \Xi_k - P_J^e \Xi)| + |(P_J^e \Xi - \Xi)| \\ & \leq |(U_H \Xi_k)(1 - P_J^e)| + \epsilon + |(\Xi)(1 - P_J^e)| \\ & \leq |(U_H \Xi_k)(1 - P_J^e)| + \epsilon + |(\Xi)(1 - P_J^e)| \leq (\sup_{k \in K} \|\Xi_k\| + \|\Xi\| + 1)\epsilon \end{aligned}$$

□

COROLLARY 3.35. — We have the following inclusion:

$$H {}_\beta \otimes_{\alpha} {}_{\mu} \mathcal{H}_\Phi \subseteq U_H(H {}_\alpha \otimes_{\hat{\beta}} {}_{\mu^o} \mathcal{H}_\Phi)$$

Proof. — By the previous corollary, we know that Ξ belongs to the weak closure of $U_H(H {}_\alpha \otimes_{\hat{\beta}} {}_{\mu^o} \mathcal{H}_\Phi)$ which is also the norm closure. Now, U_H is an isometry, that's why $U_H(H {}_\alpha \otimes_{\hat{\beta}} {}_{\mu^o} \mathcal{H}_\Phi)$ is equal to $U_H(H {}_\alpha \otimes_{\hat{\beta}} {}_{\mu^o} \mathcal{H}_\Phi)$. □

THEOREM 3.36. — $U_H : H {}_\alpha \otimes_{\hat{\beta}} {}_{\mu^o} H_\Phi \rightarrow H {}_\beta \otimes_{\alpha} {}_{\mu} H_\Phi$ is a unitary.

Proof. — By the previous corollary, we have:

$$(1) \quad H {}_\beta \otimes_{\alpha} {}_{\mu} \mathcal{H}_\Phi \subseteq U_H(H {}_\alpha \otimes_{\hat{\beta}} {}_{\mu^o} \mathcal{H}_\Phi) \subseteq U_H(H {}_\alpha \otimes_{\hat{\beta}} {}_{\mu^o} H_\Phi) \subseteq H {}_\beta \otimes_{\alpha} {}_{\mu} H_\Phi.$$

Also, using a (N^o, μ^o) -basis, we have, for all $v \in H_\Psi$ and $a \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$:

$$U_{H_\Psi}(v {}_\alpha \otimes_{\hat{\beta}} {}_{\mu^o} \Lambda_\Phi(a)) = \sum_i \xi_i {}_\beta \otimes_{\alpha} {}_{\mu} (\lambda_{\xi_i}^{\beta, \alpha})^* U_{H_\Psi}(v {}_\alpha \otimes_{\hat{\beta}} {}_{\mu^o} \Lambda_\Phi(a))$$

so that $U_{H_\Psi}(H_\Psi \underset{\mu^o}{\alpha \otimes \beta} H_\Phi) \subseteq H_\Psi \underset{\mu}{\beta \otimes \alpha} \mathcal{H}_\Phi$. The reverse inclusion is the relation (1) applied to H_Ψ . Consequently, we get:

$$U_{H_\Psi}(H_\Psi \underset{\mu^o}{\alpha \otimes \beta} H_\Phi) = U_{H_\Psi}(H_\Psi \underset{\mu^o}{\alpha \otimes \beta} \mathcal{H}_\Phi)$$

Since U_{H_Ψ} is an isometry, $H_\Psi \underset{\mu^o}{\alpha \otimes \beta} H_\Phi = H_\Psi \underset{\mu^o}{\alpha \otimes \beta} \mathcal{H}_\Phi$ and, so $\mathcal{H}_\Phi = H_\Phi$. Finally, by inclusion (1), we obtain $U_H(H \underset{\mu^o}{\alpha \otimes \beta} H_\Phi) = H \underset{\mu}{\beta \otimes \alpha} H_\Phi$. \square

COROLLARY 3.37. — *If $[F]$ denote the linear span of a subset F of a vector space E , we have:*

$$\begin{aligned} H_\Phi &= [\Lambda_\Phi((\omega_{v,w} \underset{\mu}{\beta \otimes \alpha} id)(\Gamma(a)) | v, w \in D(H_\beta, \mu^o), a \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}] \\ &= [(\lambda_w^{\beta, \alpha})^* U_H(v \underset{\mu^o}{\alpha \otimes \beta} \Lambda_\Phi(a)) | v \in H, w \in D(H_\beta, \mu^o), a \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}] \\ &= [(\omega_{v,w} * id)(U_H)\xi | v \in D(\alpha H, \mu), w \in D(H_\beta, \mu^o), \xi \in H_\Phi] \end{aligned}$$

Proof. — The second equality comes from 3.18. The last one is clear. It's sufficient to prove that the last subspace is equal to H_Φ . Let $\eta \in H_\Phi$ in the orthogonal of:

$$[(\omega_{v,w} * id)(U_H)\xi | v \in D(\alpha H, \mu), w \in D(H_\beta, \mu^o), \xi \in H_\Phi]$$

Then, for all $v \in D(\alpha H, \mu)$, $w \in D(H_\beta, \mu^o)$ and $\xi \in H_\Phi$, we have:

$$(U_H(v \underset{\mu^o}{\alpha \otimes \beta} \xi) | w \underset{\mu}{\beta \otimes \alpha} \eta) = ((\omega_{v,w} * id)(U_H)\xi | \eta) = 0$$

Since U_H is a unitary, $w \underset{\mu}{\beta \otimes \alpha} \eta = 0$ for all $w \in D(H_\beta, \mu^o)$ from which we easily deduce that $\eta = 0$ (by 2.10 for example). \square

COROLLARY 3.38. — *We have $\Gamma(m) = U_H(1 \underset{N^o}{\alpha \otimes \beta} m)U_H^*$ for all $m \in M$.*

Proof. — Straightforward thanks to unitarity of U_H and 3.25. \square

3.6. Pseudo-multiplicativity. — Let put $W = U_{H_\Phi}^*$. We have already proved commutation relations of section 3.4 and, now the aim is to prove that W is a pseudo-multiplicative unitary in the sense of M. Enock and J.M Vallin ([EV00], definition 5.6):

DEFINITION 3.39. — We call **pseudo-multiplicative unitary** over N w.r.t $\alpha, \hat{\beta}, \beta$ each unitary V from $H \underset{\mu}{\beta \otimes \alpha} H$ onto $H \underset{\mu^o}{\alpha \otimes \beta} H$ which satisfies the

following commutation relations, for all $n, m \in N$:

$$(\beta(n) \underset{N^o}{\alpha \otimes \hat{\beta}} \alpha(m))V = V(\alpha(m) \underset{N}{\beta \otimes \alpha} \hat{\beta}(n))$$

and

$$(\hat{\beta}(n) \underset{N^o}{\alpha \otimes \hat{\beta}} \beta(m))V = V(\hat{\beta}(n) \underset{N}{\beta \otimes \alpha} \beta(m))$$

and the formula:

$$\begin{aligned} (V \underset{N^o}{\alpha \otimes \hat{\beta}} 1)(\sigma_{\mu^o} \underset{N^o}{\alpha \otimes \hat{\beta}} 1)(1 \underset{N^o}{\alpha \otimes \hat{\beta}} V)\sigma_{2\mu}(1 \underset{N}{\beta \otimes \alpha} \sigma_{\mu^o})(1 \underset{N}{\beta \otimes \alpha} V) = \\ (1 \underset{N^o}{\alpha \otimes \hat{\beta}} V)(V \underset{N}{\beta \otimes \alpha} 1) \end{aligned}$$

where the first σ_{μ^o} is the flip from $H \underset{\mu^o}{\alpha \otimes \hat{\beta}} H$ onto $H \underset{\mu^o}{\hat{\beta} \otimes \alpha} H$, the second is the flip from $H \underset{\mu^o}{\alpha \otimes \beta} H$ onto $H \underset{\mu}{\beta \otimes \alpha} H$ and $\sigma_{2\mu}$ is the flip from $H \underset{\mu}{\beta \otimes \alpha} H \underset{\mu}{\hat{\beta} \otimes \alpha} H$ onto $H \underset{\mu^o}{\alpha \otimes \hat{\beta}} (H \underset{\mu}{\beta \otimes \alpha} H)$. This last flip turns around the second tensor product. Moreover, parenthesis underline the fact that the representation acts on the furthest leg.

We recall, following ([Eno02], 3.5), if we use an other n.s.f weight for the construction of relative tensor product, then canonical isomorphisms of bimodules change the pseudo-multiplicative unitary into another pseudo-multiplicative unitary. The pentagonal relation comes essentially from the co-product relation. So, we compute $(id \underset{N}{\beta \star \alpha} \Gamma) \circ \Gamma$ and $(\Gamma \underset{N}{\beta \star \alpha} id) \circ \Gamma$ in terms of U_H with the following propositions 3.42 and 3.44. Until the end of the section, \mathcal{H} is an other Hilbert space on which M acts.

LEMMA 3.40. — *We have, for all $\xi_1 \in D(\alpha\mathcal{H}, \mu)$ and $\xi'_2 \in D(H_\beta, \mu^o)$:*

$$\lambda_{\xi_1}^{\alpha, \hat{\beta}} (\lambda_{\xi'_2}^{\beta, \alpha})^* = (\lambda_{\xi'_2}^{\beta, \alpha})^* \sigma_{2\mu^o} (1 \underset{N^o}{\alpha \otimes \hat{\beta}} \sigma_\mu) \lambda_{\xi_1}^{\alpha, \hat{\beta}}$$

and:

$$U_{\mathcal{H}} \lambda_{\xi_1}^{\alpha, \hat{\beta}} (\lambda_{\xi'_2}^{\beta, \alpha})^* U_H = (\lambda_{\xi'_2}^{\beta, \alpha})^* (1 \underset{N}{\beta \otimes \alpha} U_{\mathcal{H}}) \sigma_{2\mu^o} (1 \underset{N^o}{\alpha \otimes \hat{\beta}} \sigma_\mu) (1 \underset{N^o}{\alpha \otimes \beta} U_H) \lambda_{\xi_1}^{\alpha, \beta}$$

Proof. — The first equality is easy to verify and the second one comes from the first one. \square

PROPOSITION 3.41. — *The two following equations hold:*

i) for all $\xi_1 \in D(\alpha\mathcal{H}, \mu)$, $\xi'_1 \in D(\alpha H, \mu)$, $\xi_2 \in D(\mathcal{H}_\beta, \mu^o)$, $\xi'_2 \in D(H_\beta, \mu^o)$ and $\eta_1, \eta_2 \in H_\Phi$, the scalar product of:

$$(1 \underset{N}{\beta \otimes \alpha} U_{\mathcal{H}}) \sigma_{2\mu^o} (1 \underset{N^o}{\alpha \otimes \hat{\beta}} \sigma_\mu) (1 \underset{N^o}{\alpha \otimes \beta} U_H) (\sigma_\mu \underset{N^o}{\alpha \otimes \hat{\beta}} 1) ([\xi'_1 \underset{\mu}{\beta \otimes \alpha} \xi_1] \underset{\mu^o}{\alpha \otimes \hat{\beta}} \eta_1)$$

by $\xi'_2 \underset{\mu}{\beta} \otimes_{\alpha} \xi_2 \underset{\mu}{\beta} \otimes_{\alpha} \eta_2$ is equal to $((\omega_{\xi_1, \xi_2} * id)(U_{\mathcal{H}})(\omega_{\xi'_1, \xi'_2} * id)(U_H)\eta_1|\eta_2)$.

ii) for all $a \in \mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L}$, $\xi_1 \in \mathcal{H}$ and $\xi'_1, \xi'_2 \in D(H_{\beta}, \mu^o)$, the value of:

$$(\lambda_{\xi'_2}^{\beta, \alpha})^*(1 \underset{N}{\beta} \otimes_{\alpha} U_{\mathcal{H}})\sigma_{2\mu^o}(1 \underset{N^o}{\alpha} \otimes_{\hat{\beta}} \sigma_{\mu})(1 \underset{N^o}{\alpha} \otimes_{\beta} U_H)(\sigma_{\mu} \underset{N^o}{\alpha} \otimes_{\hat{\beta}} 1)$$

on $[\xi'_1 \underset{\mu}{\beta} \otimes_{\alpha} \xi_1] \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} \Lambda_{\Phi}(a)$ is equal to:

$$U_{\mathcal{H}}(\xi_1 \underset{N^o}{\alpha} \otimes_{\hat{\beta}} \Lambda_{\Phi}((\omega_{\xi'_1, \xi'_2} \underset{\mu}{\beta} \star_{\alpha} id)(\Gamma(a))))$$

Proof. — By the previous lemma, we can compute the scalar product of i) in the following way:

$$\begin{aligned} & ((\lambda_{\xi'_2}^{\beta, \alpha})^*(1 \underset{N}{\beta} \otimes_{\alpha} U_{\mathcal{H}})\sigma_{2\mu^o}(1 \underset{N^o}{\alpha} \otimes_{\hat{\beta}} \sigma_{\mu})(1 \underset{N^o}{\alpha} \otimes_{\beta} U_H)\lambda_{\xi'_1}^{\alpha, \beta}(\xi'_1 \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} \eta_1)|\xi_2 \underset{\mu}{\beta} \otimes_{\alpha} \eta_2) \\ &= (U_{\mathcal{H}}\lambda_{\xi'_1}^{\alpha, \hat{\beta}}(\lambda_{\xi'_2}^{\beta, \alpha})^*U_H(\xi'_1 \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} \eta_1)|\xi_2 \underset{\mu}{\beta} \otimes_{\alpha} \eta_2) \\ &= ((\lambda_{\xi'_2}^{\beta, \alpha})^*U_{\mathcal{H}}(\xi_1 \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} (\omega_{\xi'_1, \xi'_2} * id)(U_H)\eta_1|\eta_2) \\ &= ((\omega_{\xi_1, \xi_2} * id)(U_{\mathcal{H}})(\omega_{\xi'_1, \xi'_2} * id)(U_H)\eta_1|\eta_2) \end{aligned}$$

Also, the second assertion comes from the previous lemma and 3.17. Let's first assume that $\xi_1 \in D({}_{\alpha}\mathcal{H}, \mu)$. Then, we compute the vector in demand:

$$\begin{aligned} & (\lambda_{\xi'_2}^{\beta, \alpha})^*(1 \underset{N}{\beta} \otimes_{\alpha} U_{\mathcal{H}})\sigma_{2\mu^o}(1 \underset{N^o}{\alpha} \otimes_{\hat{\beta}} \sigma_{\mu})(1 \underset{N^o}{\alpha} \otimes_{\beta} U_H)\lambda_{\xi'_1}^{\alpha, \beta}(\xi'_1 \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} \Lambda_{\Phi}(a)) \\ &= U_{\mathcal{H}}\lambda_{\xi'_1}^{\alpha, \hat{\beta}}(\lambda_{\xi'_2}^{\beta, \alpha})^*U_H(\xi'_1 \underset{\mu^o}{\alpha} \otimes_{\hat{\beta}} \Lambda_{\Phi}(a)) \\ &= U_{\mathcal{H}}(\xi_1 \underset{N^o}{\alpha} \otimes_{\hat{\beta}} \Lambda_{\Phi}((\omega_{\xi'_1, \xi'_2} \underset{\mu}{\beta} \star_{\alpha} id)(\Gamma(a)))) \end{aligned}$$

So, we obtain the expected equality for all $\xi_1 \in D({}_{\alpha}\mathcal{H}, \mu)$. Since the two expressions are continuous in ξ_1 , density of $D({}_{\alpha}\mathcal{H}, \mu)$ in \mathcal{H} implies that the equality is still true for all $\xi_1 \in \mathcal{H}$. \square

PROPOSITION 3.42. — For all $a, b \in \mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L}$, we have:

$$\begin{aligned} & (id \underset{\mu}{\beta} \star_{\alpha} \Gamma)(\Gamma(a))\rho_{J_{\Phi}\Lambda_{\Phi}(b)}^{\beta, \alpha} \\ &= (1 \underset{N}{\beta} \otimes_{\alpha} (1 \underset{N}{\beta} \otimes_{\alpha} J_{\Phi}bJ_{\Phi})U_{\mathcal{H}})\sigma_{2\mu^o}(1 \underset{N^o}{\alpha} \otimes_{\hat{\beta}} \sigma_{\mu})(1 \underset{N^o}{\alpha} \otimes_{\beta} U_H)(\sigma_{\mu} \underset{N^o}{\alpha} \otimes_{\hat{\beta}} 1)\rho_{\Lambda_{\Phi}(a)}^{\alpha, \hat{\beta}} \end{aligned}$$

Proof. — Let $\xi_1 \in \mathcal{H}$ and $\xi'_1, \xi'_2 \in D(H_\beta, \mu^o)$. We compose the second term of the equality on the left by $(\lambda_{\xi'_2}^{\beta, \alpha})^*$ and we get:

$$(1_{\beta \otimes_\alpha J_\Phi b J_\Phi})(\lambda_{\xi'_2}^{\beta, \alpha})^* (1_{\beta \otimes_\alpha U_{\mathcal{H}}}) \sigma_{2\mu^o} (1_{\alpha \otimes_{\hat{\beta}} \sigma_\mu}) (1_{\alpha \otimes_{\beta} U_H}) (\sigma_\mu \alpha \otimes_{\hat{\beta}} 1) \rho_{\Lambda_\Phi(a)}^{\alpha, \hat{\beta}}$$

which we evaluate on $\xi'_1 \beta \otimes_\alpha \xi_1$, to get, by the previous proposition and 3.16:

$$\begin{aligned} & (1_{\beta \otimes_\alpha J_\Phi b J_\Phi}) U_{\mathcal{H}} (\xi_1 \alpha \otimes_{\hat{\beta}} \Lambda_\Phi((\omega_{\xi'_1, \xi'_2} \beta \star_\alpha id)(\Gamma(a)))) \\ &= \Gamma((\omega_{\xi'_1, \xi'_2} \beta \star_\alpha id)(\Gamma(a))) \rho_{J_\Phi \Lambda_\Phi(b)}^{\beta, \alpha} \xi_1 \\ &= (\lambda_{\xi'_2}^{\beta, \alpha})^* (id \beta \star_\alpha \Gamma)(\Gamma(a)) \rho_{J_\Phi \Lambda_\Phi(b)}^{\beta, \alpha} (\xi'_1 \beta \otimes_\alpha \xi_1) \end{aligned}$$

So, the proposition holds. \square

LEMMA 3.43. — For all $X \in M \beta \star_\alpha M \subset (1_{\beta \otimes_\alpha \hat{\beta}}(N))'$, we have:

$$(\Gamma \beta \star_\alpha id)(X) = (U_H \beta \otimes_\alpha 1) (1_{\alpha \otimes_{\hat{\beta}} X}) (U_H^* \beta \otimes_\alpha 1)$$

Proof. — By 3.38, Γ is implemented by U_H so that we easily deduce the lemma. \square

PROPOSITION 3.44. — For all $a, b \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$, we have:

$$\begin{aligned} & (\Gamma \beta \star_\alpha id)(\Gamma(a)) \rho_{J_\Phi \Lambda_\Phi(b)}^{\beta, \alpha} \\ &= (1_{\beta \otimes_\alpha 1} \beta \otimes_\alpha J_\Phi b J_\Phi) (U_H \beta \otimes_\alpha 1) (1_{\alpha \otimes_{\hat{\beta}} W^*}) (U_H^* \alpha \otimes_{\hat{\beta}} 1) \rho_{\Lambda_\Phi(a)}^{\alpha, \hat{\beta}} \end{aligned}$$

Proof. — By the previous lemma and 3.16, we can compute:

$$\begin{aligned} & (1_{\beta \otimes_\alpha 1} \beta \otimes_\alpha J_\Phi b J_\Phi) (U_H \beta \otimes_\alpha 1) (1_{\alpha \otimes_{\hat{\beta}} W^*}) (U_H^* \alpha \otimes_{\hat{\beta}} 1) \rho_{\Lambda_\Phi(a)}^{\alpha, \hat{\beta}} \\ &= (U_H \beta \otimes_\alpha 1) (1_{\alpha \otimes_{\hat{\beta}} 1} \beta \otimes_\alpha J_\Phi b J_\Phi) (1_{\alpha \otimes_{\hat{\beta}} W^*}) \rho_{\Lambda_\Phi(a)}^{\alpha, \hat{\beta}} U_H^* \\ &= (U_H \beta \otimes_\alpha 1) (1_{\alpha \otimes_{\hat{\beta}} (1_{\beta \otimes_\alpha J_\Phi b J_\Phi}) W^*} \rho_{\Lambda_\Phi(a)}^{\alpha, \hat{\beta}}) U_H^* \\ &= (U_H \beta \otimes_\alpha 1) (1_{\alpha \otimes_{\hat{\beta}} \Gamma(a)} \rho_{J_\Phi \Lambda_\Phi(b)}^{\beta, \alpha}) U_H^* \\ &= (U_H \beta \otimes_\alpha 1) (1_{\alpha \otimes_{\hat{\beta}} \Gamma(a)}) (U_H^* \alpha \otimes_{\hat{\beta}} 1) \rho_{J_\Phi \Lambda_\Phi(b)}^{\beta, \alpha} = (\Gamma \beta \star_\alpha id)(\Gamma(a)) \rho_{J_\Phi \Lambda_\Phi(b)}^{\beta, \alpha} \end{aligned}$$

\square

COROLLARY 3.45. — *The following relation is satisfied:*

$$\begin{aligned} & (U_H^* \underset{N^o}{\alpha \otimes \hat{\beta}} 1)(\sigma_{\mu^o} \underset{N^o}{\alpha \otimes \hat{\beta}} 1)(1 \underset{N^o}{\alpha \otimes \hat{\beta}} U_H^*) \sigma_{2\mu}(1 \underset{N}{\beta \otimes \alpha} \sigma_{\mu^o})(1 \underset{N}{\beta \otimes \alpha} W) \\ & = (1 \underset{N^o}{\alpha \otimes \hat{\beta}} W)(U_H^* \underset{N}{\beta \otimes \alpha} 1) \end{aligned}$$

Proof. — We put together 3.42 (with $\mathcal{H} = H_\Phi$) and 3.44 thanks to the co-product relation. We get:

$$\begin{aligned} & (1 \underset{N}{\beta \otimes \alpha} W^*) \sigma_{2\mu^o}(1 \underset{N^o}{\alpha \otimes \hat{\beta}} \sigma_\mu)(1 \underset{N^o}{\alpha \otimes \beta} U_H) \\ & = (U_H \underset{N}{\beta \otimes \alpha} 1)(1 \underset{N^o}{\alpha \otimes \hat{\beta}} W^*)(U_H^* \underset{N^o}{\alpha \otimes \hat{\beta}} 1)(\sigma_{\mu^o} \underset{N^o}{\alpha \otimes \hat{\beta}} 1) \end{aligned}$$

Take adjoint and we are. \square

THEOREM 3.46. — *W is a pseudo-multiplicative unitary over N w.r.t $\alpha, \hat{\beta}, \beta$.*

Proof. — W is a unitary from $H_\Phi \underset{\mu}{\beta \otimes \alpha} H_\Phi$ onto $H_\Phi \underset{\mu^o}{\alpha \otimes \hat{\beta}} H_\Phi$ which satisfies the four required commutation relations. The previous corollary, with $H = H_\Phi$, finishes the proof. \square

Similar results hold for the right version:

THEOREM 3.47. — *If $W' = U'_{H_\Psi}$, then the following relation makes sense and holds:*

$$\begin{aligned} & (W' \underset{N}{\beta \otimes \alpha} 1)(\sigma_\mu \underset{N}{\beta \otimes \alpha} 1)(1 \underset{N}{\beta \otimes \alpha} U'_H) \sigma_{2\mu^o}(1 \underset{N^o}{\hat{\alpha} \otimes \beta} \sigma_\mu)(1 \underset{N^o}{\hat{\alpha} \otimes \beta} U'_H) \\ & = (1 \underset{N}{\beta \otimes \alpha} U'_H)(W' \underset{N^o}{\hat{\alpha} \otimes \beta} 1) \end{aligned}$$

If $H = H_\Psi$, then W' is a pseudo-multiplicative unitary over N^o w.r.t $\beta, \alpha, \hat{\alpha}$.

Proof. — For example, it is sufficient to apply the previous results with the opposite Hopf bimodule. \square

3.7. Von Neumann algebra generated by right leg of the fundamental unitary. —

DEFINITION 3.48. — We call $A(U'_H)$ (resp. $\mathcal{A}(U'_H)$) the weak closure in $\mathcal{L}(H)$ of the vector space (resp. von Neumann algebra) generated by $(\omega_{v,w} * id)(U'_H)$ with $v \in D(\hat{\alpha}(H_\Psi), \mu)$ and $w \in D((H_\Psi)_\beta, \mu^o)$.

PROPOSITION 3.49. — $A(U'_H)$ is a non-degenerate involutive algebra i.e $A(U'_H) = \mathcal{A}(U'_H)$ such that:

$$\alpha(N) \cup \beta(N) \subseteq A(U'_H) = \mathcal{A}(U'_H) \subseteq M \subseteq \hat{\alpha}(N)'$$

Moreover, we have:

$$x \in \mathcal{A}(U'_H)' \cap \mathcal{L}(H) \iff U'_H(1_{\hat{\alpha} \otimes_{\beta} x}) = (1_{\beta \otimes_{\alpha} x})U'_H$$

In fact, we will see later that $A(U'_H) = \mathcal{A}(U'_H) = M$.

Proof. — The second and third points are obtained in [EV00] (theorem 6.1). As far as the first point is concerned, it comes from [Eno02] (proposition 3.6) and 3.20 which proves that $A(U'_H)$ is involutive. \square

To summarize the results of this section, we state the following theorem:

THEOREM 3.50. — Let $(N, M, \alpha, \beta, \Gamma)$ be a Hopf bimodule, T_L (resp. T_R) be a left (resp. right) invariant n.s.f operator-valued weight. Then, for all n.s.f weight μ on N , if $\Phi = \mu \circ \alpha^{-1} \circ T_L$, then the application:

$$v_{\alpha \otimes_{\hat{\beta}} \mu^o} \Lambda_{\Phi}(a) \mapsto \sum_{i \in I} \xi_i_{\beta \otimes_{\alpha} \mu} \Lambda_{\Phi}((\omega_{v, \xi_i} \beta \star_{\alpha} id)(\Gamma(a)))$$

for all $v \in D((H_{\Phi})_{\beta}, \mu^o)$, $a \in \mathcal{N}_{T_L} \cap \mathcal{N}_{\Phi}$, (N^o, μ^o) -basis $(\xi_i)_{i \in I}$ of $(H_{\Phi})_{\beta}$ and where $\hat{\beta}(n) = J_{\Phi} \alpha(n^*) J_{\Phi}$, extends to a unitary W , the adjoint of which W^* is a pseudo-multiplicative unitary over N w.r.t $\alpha, \hat{\beta}, \beta$ from $H_{\Phi} \alpha \otimes_{\hat{\beta}} H_{\Phi}$ onto $H_{\Phi} \beta \otimes_{\alpha} H_{\Phi}$.

Moreover, for all $m \in M$, we have:

$$\Gamma(m) = W^*(1_{\alpha \otimes_{\hat{\beta}} m})W$$

Also, we have similar results from T_R .

4. Antipode

Until the end, we introduce a new natural hypothesis which gives a link between the right (resp. left) invariant operator-valued weight and the (resp. anti-) representation of the basis. Then we construct a closed antipode with polar decomposition which leads to a co-involution and a one-parameter group of automorphisms of M called scaling group.

4.1. Measured quantum groupoid's definition. —

DEFINITION 4.1. — We say that a n.s.f operator-valued weight T_L from M to $\alpha(N)$ is **β -adapted** if there exists a n.s.f weight ν_L on N such that:

$$\sigma_t^{T_L}(\beta(n)) = \beta(\sigma_{-t}^{\nu_L}(n))$$

for all $n \in N$ and $t \in \mathbb{R}$. We also say that T_L is β -adapted w.r.t ν_L .

We say that a n.s.f operator-valued weight T_R from M to $\beta(N)$ is **α -adapted** if there exists a n.s.f weight ν_R on N such that:

$$\sigma_t^{T_R}(\alpha(n)) = \alpha(\sigma_t^{\nu_R}(n))$$

for all $n \in N$ and $t \in \mathbb{R}$. We also say that T_R is α -adapted w.r.t ν_R .

DEFINITION 4.2. — A Hopf bimodule $(N, M, \alpha, \beta, \Gamma)$ with left (resp. right) invariant n.s.f operator-valued weight T_L (resp. T_R) from M to $\alpha(N)$ (resp. $\beta(N)$) is said to be a **measured quantum groupoid** if there exists a n.s.f weight ν on N such that T_L is β -adapted w.r.t ν and T_R is α -adapted w.r.t ν . Then, we denote by $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$ the measured quantum groupoid and we say that ν is **quasi-invariant**.

REMARK 4.3. — If a n.s.f operator-valued weight T_L from M to $\alpha(N)$ is β -adapted w.r.t ν and if R is a co-involution of M , then the n.s.f operator-valued weight $R \circ T_L \circ R$ from M to $\beta(N)$ is α -adapted w.r.t the same weight ν .

LEMMA 4.4. — If μ is a n.s.f weight on N and if an operator-valued weight T_L is β -adapted w.r.t ν , then there exists an operator-valued weight S^μ from M to $\beta(N)$, which is α -adapted w.r.t μ such that $\mu \circ \alpha^{-1} \circ T_L = \nu \circ \beta^{-1} \circ S^\mu$. Also, if χ is a n.s.f weight on N and if an operator-valued weight T_R is α -adapted w.r.t ν , then there exists an operator-valued weight S_χ from M to $\alpha(N)$ normal, which is β -adapted w.r.t χ such that $\chi \circ \beta^{-1} \circ T_R = \nu \circ \beta^{-1} \circ S_\chi$.

Proof. — For all $n \in N$ and $t \in \mathbb{R}$, we have $\sigma_t^{\mu \circ \alpha^{-1} \circ T_L}(\beta(n)) = \sigma_t^{\nu \circ \beta^{-1}}(\beta(n))$. By Haagerup's theorem, we obtain the existence of S^μ which is clearly adapted. The second part of the lemma is very similar. \square

Let $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$ be a measured quantum groupoid. Then the opposite measured quantum groupoid is $(N^o, M, \beta, \alpha, \varsigma_N \circ \Gamma, \nu^o, T_R, T_L)$. We put:

$$\Phi = \nu \circ \alpha^{-1} \circ T_L \quad \text{and} \quad \Psi = \nu \circ \beta^{-1} \circ T_R$$

We also put $S^\nu = S_L$ and $S_\nu = S_R$. By 3.6 and 3.8, we have:

$$\Lambda_\Phi(\mathcal{T}_{\Phi, S_L}) \subseteq J_\Phi \Lambda_\Phi(\mathcal{N}_\Phi \cap \mathcal{N}_{S_L}) \subseteq D((H_\Phi)_\beta, \nu^o)$$

and we have $R^{\beta, \nu^o}(J_\Phi \Lambda_\Phi(a)) = J_\Phi \Lambda_{S_L}(a) J_\nu$ for all $a \in \mathcal{N}_\Phi \cap \mathcal{N}_{S_L}$.

4.2. The operator G . — We construct now an closed unbounded operator on H_Φ with polar decomposition which gives needed elements to construct the antipode. We have the following lemmas:

LEMMA 4.5. — *For all $\lambda \in \mathbb{C}$, $x \in \mathcal{D}(\sigma_{i\lambda}^\nu)$ and $\xi, \xi' \in \Lambda_\Phi(\mathcal{T}_{\Phi, T_L})$, we have:*

$$(2) \quad \begin{aligned} \alpha(x)\Delta_\Phi^\lambda &\subseteq \Delta_\Phi^\lambda \alpha(\sigma_{i\lambda}^\nu(x)) \\ R^{\alpha, \nu}(\Delta_\Phi^\lambda \xi)\Delta_\nu^\lambda &\subseteq \Delta_\Phi^\lambda R^{\alpha, \nu}(\xi) \\ \text{and } \sigma_{i\lambda}^\nu(< \Delta_\Phi^\lambda \xi, \xi' >_{\alpha, \nu}) &= < \xi, \Delta_\Phi^{\bar{\lambda}} \xi' >_{\alpha, \nu} \end{aligned}$$

and:

$$(3) \quad \begin{aligned} \hat{\beta}(x)\Delta_\Phi^\lambda &\subseteq \Delta_\Phi^\lambda \hat{\beta}(\sigma_{i\lambda}^\nu(x)) \\ R^{\hat{\beta}, \nu^o}(\Delta_\Phi^\lambda \xi)\Delta_\nu^\lambda &\subseteq \Delta_\Phi^\lambda R^{\hat{\beta}, \nu^o}(\xi) \\ \text{and } \sigma_{i\lambda}^\nu(< \Delta_\Phi^\lambda \xi, \xi' >_{\hat{\beta}, \nu^o}) &= < \xi, \Delta_\Phi^{\bar{\lambda}} \xi' >_{\hat{\beta}, \nu^o} . \end{aligned}$$

Proof. — Straightforward. \square

LEMMA 4.6. — [Sau86] *We can define, for all $\lambda \in \mathbb{C}$, a closed operator $\Delta_\Phi^\lambda \underset{\nu^o}{\alpha \otimes \hat{\beta}} \Delta_\Phi^\lambda$ which naturally acts on elementary tensor products.*

Proof. — Let $\lambda \in \mathbb{C}$. We define a linear operator Δ_λ on the algebraic tensor product $\Lambda_\Phi(\mathcal{T}_{\Phi, S_L}) \odot \mathcal{D}(\Delta_\Phi^\lambda) \subseteq D({}_\alpha H_\Phi, \nu) \odot H_\Phi$ by the formula:

$$\Delta_\lambda(\xi \odot \eta) = \Delta_\Phi^\lambda \xi \odot \Delta_\Phi^\lambda \eta$$

For all $\xi, \xi' \in \Lambda_\Phi(\mathcal{T}_{\Phi, S_L})$, $\eta \in \mathcal{D}(\Delta_\Phi^\lambda)$ and $\eta' \in \mathcal{D}(\Delta_\Phi^{\bar{\lambda}})$, relations (2) imply:

$$\begin{aligned} (\Delta_\lambda(\xi \odot \eta)|\xi' \odot \eta') &= (\hat{\beta}(< \Delta_\Phi^\lambda \xi, \xi' >_{\alpha, \nu})\Delta_\Phi^\lambda \eta|\eta') \\ &= (\Delta_\Phi^\lambda \hat{\beta}(\sigma_{i\lambda}^\nu(< \Delta_\Phi^\lambda \xi, \xi' >_{\alpha, \nu}))\eta|\eta') \\ &= (\hat{\beta}(< \xi, \Delta_\Phi^{\bar{\lambda}} \xi' >_{\alpha, \nu})\eta|\Delta_\Phi^{\bar{\lambda}} \eta') \\ &= (\xi \underset{\nu^o}{\alpha \otimes \hat{\beta}} \eta|\Delta_\Phi^{\bar{\lambda}} \xi' \underset{\nu}{\alpha \otimes \hat{\beta}} \Delta_\Phi^{\bar{\lambda}} \eta') \end{aligned}$$

So, for all $\lambda \in \mathbb{C}$, Δ_λ go through quotient to give a densely defined operator $\tilde{\Delta}_\lambda$ on $H_\Phi \underset{\nu^o}{\alpha \otimes \hat{\beta}} H_\Phi$. Moreover, previous equalities prove that $\tilde{\Delta}_{\bar{\lambda}} \subseteq \tilde{\Delta}_\lambda^*$. So, $\tilde{\Delta}_\lambda^*$ is densely defined which means $\tilde{\Delta}_\lambda$ is closable. The operator, we look for, is this closure. [Sau86] gives full ideas about the subject. \square

Since, for all $x \in N$, we have $J_\Phi \alpha(x) = \hat{\beta}(x^*) J_\Phi$, by [Sau83a], we can define a unitary anti-linear operator:

$$J_\Phi \underset{\nu^o}{\alpha \otimes \hat{\beta}} J_\Phi : H_\Phi \underset{\nu^o}{\alpha \otimes \hat{\beta}} H_\Phi \rightarrow H_\Phi \underset{\nu}{\hat{\beta} \otimes \alpha} H_\Phi$$

such that the adjoint is $J_\Phi \underset{\nu}{\hat{\beta} \otimes \alpha} J_\Phi$. Also, by composition, it is possible to define a natural closed anti-linear operator:

$$S_\Phi \underset{\nu^o}{\alpha \otimes \hat{\beta}} S_\Phi : H_\Phi \underset{\nu^o}{\alpha \otimes \hat{\beta}} H_\Phi \rightarrow H_\Phi \underset{\nu}{\hat{\beta} \otimes \alpha} H_\Phi$$

In the same way, if $F_\Phi = S_\Phi^*$, then it is possible to define a natural closed anti-linear operator: $F_\Phi \underset{\nu}{\hat{\beta} \otimes \alpha} F_\Phi : H_\Phi \underset{\nu}{\hat{\beta} \otimes \alpha} H_\Phi \rightarrow H_\Phi \underset{\nu^o}{\alpha \otimes \hat{\beta}} H_\Phi$ and we have:

$$(S_\Phi \underset{\nu^o}{\alpha \otimes \hat{\beta}} S_\Phi)^* = F_\Phi \underset{\nu}{\hat{\beta} \otimes \alpha} F_\Phi$$

LEMMA 4.7. — *For all $c \in (\mathcal{N}_\Phi \cap \mathcal{N}_{T_L})^*(\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})$, $e \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$ and all net $(e_k)_{k \in K}$ of elements of $\mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$ weakly converging to 1, then $(\lambda_{J_\Psi \Lambda_\Psi(e_k)}^{\beta, \alpha})^* (1 \underset{\nu}{\beta \otimes \alpha} J_\Phi e J_\Phi) U_{H_\Psi} \rho_{\Lambda_\Phi(c^*)}^{\alpha, \hat{\beta}}$ converges to $(\lambda_{\Lambda_\Psi(c)}^{\hat{\alpha}, \beta})^* U_{H_\Phi}^* \rho_{J_\Phi \Lambda_\Phi(e)}^{\beta, \alpha}$ in the weak topology.*

Proof. — By 3.16, we have, for all $k \in K$:

$$\begin{aligned} & (\lambda_{J_\Psi \Lambda_\Psi(e_k)}^{\beta, \alpha})^* (1 \underset{\nu}{\beta \otimes \alpha} J_\Phi e J_\Phi) U_{H_\Psi} \rho_{\Lambda_\Phi(c^*)}^{\alpha, \hat{\beta}} \\ &= (\lambda_{J_\Psi \Lambda_\Psi(e_k)}^{\beta, \alpha})^* \Gamma(c^*) \rho_{J_\Phi \Lambda_\Phi(e)}^{\beta, \alpha} = \left(\Gamma(c) \lambda_{J_\Psi \Lambda_\Psi(e_k)}^{\beta, \alpha} \right)^* \rho_{J_\Phi \Lambda_\Phi(e)}^{\beta, \alpha} \\ &= \left((J_\Psi e_k J_\Psi \underset{N}{\beta \otimes \alpha} 1) U_{H_\Phi}' \lambda_{\Lambda_\Psi(c)}^{\hat{\alpha}, \beta} \right)^* \rho_{J_\Phi \Lambda_\Phi(e)}^{\beta, \alpha} \\ &= (\lambda_{\Lambda_\Psi(c)}^{\hat{\alpha}, \beta})^* U_{H_\Phi}^{t*} (J_\Psi e_k^* J_\Psi \underset{N}{\beta \otimes \alpha} 1) \rho_{J_\Phi \Lambda_\Phi(e)}^{\beta, \alpha} = (\lambda_{\Lambda_\Psi(c)}^{\hat{\alpha}, \beta})^* U_{H_\Phi}^{t*} \rho_{J_\Phi \Lambda_\Phi(e)}^{\beta, \alpha} J_\Psi e_k^* J_\Psi \end{aligned}$$

This computation implies the lemma. \square

LEMMA 4.8. — *If $c \in (\mathcal{N}_\Phi \cap \mathcal{N}_{T_L})^*(\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})$, $e \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$, $\eta \in H_\Psi$, $v \in H_\Phi$ and a net $(e_k)_{k \in K}$ of $\mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$ converges weakly to 1, then the net:*

$$((U_{H_\Psi}(\eta \underset{\nu^o}{\alpha \otimes \hat{\beta}} \Lambda_\Phi(c^*)) | J_\Psi \Lambda_\Psi(e_k) \underset{\nu}{\beta \otimes \alpha} J_\Phi e^* J_\Phi v))_{k \in K}$$

converges to $(\eta | (\rho_{J_\Phi \Lambda_\Phi(e)}^{\beta, \alpha})^ U_{H_\Phi}' (\Lambda_\Psi(c) \underset{\nu^o}{\alpha \otimes \hat{\beta}} v))$.*

Proof. — It's a re-formulation of the previous lemma. \square

PROPOSITION 4.9. — *Let $(\eta_i)_{i \in I}$ be a (N, ν) -basis of ${}_{\alpha}H$, $\Xi \in H_{\Psi} \beta \otimes_{\alpha} H$, $u \in D({}_{\alpha}H, \nu)$, $c \in (\mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L})^*(\mathcal{N}_{\Psi} \cap \mathcal{N}_{T_R})$, $h \in \mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L}$ and e be an element of $\mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L} \cap \mathcal{N}_{\Phi}^* \cap \mathcal{N}_{T_L}^*$. Then, we have:*

$$\lim_k \sum_{i \in I} (\eta_i \alpha \otimes_{\beta} h^*(\lambda_{J_{\Phi} \Lambda_{\Phi}(e_k)}^{\beta, \alpha})^* U_{H_{\Psi}}((\rho_{\eta_i}^{\beta, \alpha})^* \Xi \alpha \otimes_{\beta} \Lambda_{\Phi}(c^*)) | u \alpha \otimes_{\beta} J_{\Phi} \Lambda_{\Phi}(e^*))$$

exists and is equal to $((\rho_u^{\beta, \alpha})^* \Xi | (\rho_{J_{\Phi} \Lambda_{\Phi}(e)}^{\beta, \alpha})^* U'_{H_{\Psi}}(\Lambda_{\Psi}(c) \alpha \otimes_{\beta} \Lambda_{\Phi}(h)))$.

Proof. — By 3.21 and 3.22, we can compute, for all $i \in I$ and $k \in K$:

$$\begin{aligned} & (\eta_i \alpha \otimes_{\beta} h^*(\lambda_{J_{\Phi} \Lambda_{\Phi}(e_k)}^{\beta, \alpha})^* U_{H_{\Psi}}((\rho_{\eta_i}^{\beta, \alpha})^* \Xi \alpha \otimes_{\beta} \Lambda_{\Phi}(c^*)) | u \alpha \otimes_{\beta} J_{\Phi} \Lambda_{\Phi}(e^*)) \\ &= (\hat{\beta}(< \eta_i, u >_{\alpha, \nu}) (\lambda_{J_{\Phi} \Lambda_{\Phi}(e_k)}^{\beta, \alpha})^* U_{H_{\Psi}}((\rho_{\eta_i}^{\beta, \alpha})^* \Xi \alpha \otimes_{\beta} \Lambda_{\Phi}(c^*)) | g J_{\Phi} \Lambda_{\Phi}(e^*)) \\ &= ((\lambda_{J_{\Phi} \Lambda_{\Phi}(e_k)}^{\beta, \alpha})^* (1_{\beta} \otimes_{\alpha} \hat{\beta}(< \eta_i, u >_{\alpha, \nu}) U_{H_{\Psi}}((\rho_{\eta_i}^{\beta, \alpha})^* \Xi \alpha \otimes_{\beta} \Lambda_{\Phi}(c^*)) | J_{\Phi} e^* J_{\Phi} \Lambda_{\Phi}(h))) \\ &= ((\lambda_{J_{\Phi} \Lambda_{\Phi}(e_k)}^{\beta, \alpha})^* U_{H_{\Psi}}(\beta(< \eta_i, u >_{\alpha, \nu}) (\rho_{\eta_i}^{\beta, \alpha})^* \Xi \alpha \otimes_{\beta} \Lambda_{\Phi}(c^*)) | J_{\Phi} e^* J_{\Phi} \Lambda_{\Phi}(h)) \end{aligned}$$

Take the sum over i to obtain:

$$\begin{aligned} & \sum_{i \in I} (\eta_i \alpha \otimes_{\beta} h^*(\lambda_{J_{\Phi} \Lambda_{\Phi}(e_k)}^{\beta, \alpha})^* U_{H_{\Psi}}((\rho_{\eta_i}^{\beta, \alpha})^* \Xi \alpha \otimes_{\beta} \Lambda_{\Phi}(c^*)) | u \alpha \otimes_{\beta} J_{\Phi} \Lambda_{\Phi}(e^*)) \\ &= (U_{H_{\Psi}}((\rho_u^{\beta, \alpha})^* \Xi \alpha \otimes_{\beta} \Lambda_{\Phi}(c^*)) | J_{\Phi} \Lambda_{\Phi}(e_k) \beta \otimes_{\alpha} J_{\Phi} e^* J_{\Phi} \Lambda_{\Phi}(h)) \end{aligned}$$

so that lemma 4.8 implies:

$$\begin{aligned} & \lim_k \sum_{i \in I} (\eta_i \alpha \otimes_{\beta} h^*(\lambda_{J_{\Phi} \Lambda_{\Phi}(e_k)}^{\beta, \alpha})^* U_{H_{\Psi}}((\rho_{\eta_i}^{\beta, \alpha})^* \Xi \alpha \otimes_{\beta} \Lambda_{\Phi}(c^*)) | u \alpha \otimes_{\beta} J_{\Phi} \Lambda_{\Phi}(e^*)) \\ &= ((\rho_u^{\beta, \alpha})^* \Xi | (\rho_{J_{\Phi} \Lambda_{\Phi}(e)}^{\beta, \alpha})^* U'_{H_{\Psi}}(\Lambda_{\Psi}(c) \alpha \otimes_{\beta} \Lambda_{\Phi}(h))) \end{aligned}$$

□

PROPOSITION 4.10. — *For all $a, c \in (\mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L})^*(\mathcal{N}_{\Psi} \cap \mathcal{N}_{T_R})$, $b, d \in \mathcal{T}_{\Psi, T_R}$ and $g, h \in \mathcal{T}_{\Phi, S_L}$, the following vector:*

$$U_{H_{\Phi}}^* \Gamma(g^*)(\Lambda_{\Phi}(h) \beta \otimes_{\alpha} (\lambda_{\Lambda_{\Psi}(\sigma_{\Psi_i}(b^*))}^{\beta, \alpha})^* U_{H_{\Psi}}(\Lambda_{\Psi}(a) \alpha \otimes_{\beta} \Lambda_{\Phi}((cd)^*)))$$

belongs to $\mathcal{D}(S_{\Phi} \alpha \otimes_{\beta} S_{\Phi})$ and the value of $\sigma_{\nu}(S_{\Phi} \alpha \otimes_{\beta} S_{\Phi})$ on this vector is equal to:

$$U_{H_{\Phi}}^* \Gamma(h^*)(\Lambda_{\Phi}(g) \beta \otimes_{\alpha} (\lambda_{\Lambda_{\Psi}(\sigma_{\Psi_i}(d^*))}^{\beta, \alpha})^* U_{H_{\Psi}}(\Lambda_{\Psi}(c) \alpha \otimes_{\beta} \Lambda_{\Phi}((ab)^*)))$$

Proof. — For the proof, let denote by $\Xi_1 = U'_{H_\Phi}(\Lambda_\Psi(ab) \underset{\nu^o}{\hat{\alpha}} \otimes_\beta \Lambda_\Phi(h))$ and by $\Xi_2 = U'_{H_\Phi}(\Lambda_\Psi(cd) \underset{\nu^o}{\hat{\alpha}} \otimes_\beta \Lambda_\Phi(g))$. Then, for all $e, f \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}^* \cap \mathcal{N}_\Phi^*$, the scalar product of $F_\Phi J_\Phi \Lambda_\Phi(e^*) \underset{\nu^o}{\hat{\alpha}} \otimes_\beta F_\Phi J_\Phi \Lambda_\Phi(f)$ by:

$$U_{H_\Phi}^* \Gamma(g^*)(\Lambda_\Phi(h) \underset{\nu}{\beta} \otimes_\alpha (\lambda_{\Lambda_\Psi(\sigma_{-i}^{\Psi_i}(b^*))}^{\beta, \alpha})^* U_{H_\Psi}(\Lambda_\Psi(a) \underset{\nu^o}{\hat{\alpha}} \otimes_\beta \Lambda_\Phi((cd)^*)))$$

is equal to the scalar product of $J_\Phi \Lambda_\Phi(e) \underset{\nu^o}{\hat{\alpha}} \otimes_\beta J_\Phi \Lambda_\Phi(f^*)$ by:

$$U_{H_\Phi}^* \Gamma(g^*)(\Lambda_\Phi(h) \underset{\nu}{\beta} \otimes_\alpha (\lambda_{\Lambda_\Psi(\sigma_{-i}^{\Psi_i}(b^*))}^{\beta, \alpha})^* U_{H_\Psi}(\Lambda_\Psi(a) \underset{\nu^o}{\hat{\alpha}} \otimes_\beta \Lambda_\Phi((cd)^*)))$$

By 3.34, this scalar product is equal to the limit over k of the sum over i of:

$$(J_\Phi \Lambda_\Phi(e) \underset{\nu^o}{\hat{\alpha}} \otimes_\beta J_\Phi \Lambda_\Phi(f^*) | \eta_i \underset{\nu^o}{\hat{\alpha}} \otimes_\beta g^*(\lambda_{J_\Psi \Lambda_\Psi(e_k)}^{\beta, \alpha})^* U_{H_\Psi}((\rho_{\eta_i}^{\beta, \alpha})^* \Xi_1 \underset{\nu^o}{\hat{\alpha}} \otimes_\beta \Lambda_\Phi((cd)^*)))$$

By the previous proposition applied with $\Xi = \Xi_1$, we get the symmetric expression:

$$((\rho_{J_\Phi \Lambda_\Phi(f)}^{\beta, \alpha})^* \Xi_2 | (\rho_{J_\Phi \Lambda_\Phi(e)}^{\beta, \alpha})^* \Xi_1)$$

so that, again by the previous proposition applied, this time, with $\Xi = \Xi_2$ we obtain the limit over k of the sum over i of:

$$(\eta_i \underset{\nu^o}{\hat{\alpha}} \otimes_\beta h^*(\lambda_{J_\Psi \Lambda_\Psi(e_k)}^{\beta, \alpha})^* U_{H_\Psi}((\rho_{\eta_i}^{\beta, \alpha})^* \Xi_2 \underset{\nu^o}{\hat{\alpha}} \otimes_\beta \Lambda_\Phi((ab)^*)) | J_\Phi \Lambda_\Phi(f) \underset{\nu^o}{\hat{\alpha}} \otimes_\beta J_\Phi \Lambda_\Phi(e^*))$$

This last expression is equal to the scalar product of:

$$U_{H_\Phi}^* \Gamma(h^*)(\Lambda_\Phi(g) \underset{\nu}{\beta} \otimes_\alpha (\lambda_{\Lambda_\Psi(\sigma_{-i}^{\Psi_i}(d^*))}^{\beta, \alpha})^* U_{H_\Psi}(\Lambda_\Psi(c) \underset{\nu^o}{\hat{\alpha}} \otimes_\beta \Lambda_\Phi((ab)^*)))$$

by $J_\Phi \Lambda_\Phi(f) \underset{\nu^o}{\hat{\alpha}} \otimes_\beta J_\Phi \Lambda_\Phi(e^*)$ and to the scalar product of:

$$\sigma_{\nu^o} U_{H_\Phi}^* \Gamma(h^*)(\Lambda_\Phi(g) \underset{\nu}{\beta} \otimes_\alpha (\lambda_{\Lambda_\Psi(\sigma_{-i}^{\Psi_i}(d^*))}^{\beta, \alpha})^* U_{H_\Psi}(\Lambda_\Psi(c) \underset{\nu^o}{\hat{\alpha}} \otimes_\beta \Lambda_\Phi((ab)^*)))$$

by $J_\Phi \Lambda_\Phi(e^*) \underset{\beta}{\hat{\alpha}} \otimes_\alpha J_\Phi \Lambda_\Phi(f)$. Since the linear span of $J_\Phi \Lambda_\Phi(e^*) \underset{\beta}{\hat{\alpha}} \otimes_\alpha J_\Phi \Lambda_\Phi(f)$

where $e, f \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}^* \cap \mathcal{N}_\Phi^*$ is a core of $F_\Phi \underset{\beta}{\hat{\alpha}} \otimes_\alpha F_\Phi$, we get that:

$$U_{H_\Phi}^* \Gamma(g^*)(\Lambda_\Phi(h) \underset{\nu}{\beta} \otimes_\alpha (\lambda_{\Lambda_\Psi(\sigma_{-i}^{\Psi_i}(b^*))}^{\beta, \alpha})^* U_{H_\Psi}^*(\Lambda_\Psi(a) \underset{\nu^o}{\hat{\alpha}} \otimes_\beta \Lambda_\Phi((cd)^*)))$$

belongs to $\mathcal{D}(S_\Phi \underset{\nu^o}{\hat{\alpha}} \otimes_\beta S_\Phi)$ and the value of $S_\Phi \underset{\nu^o}{\hat{\alpha}} \otimes_\beta S_\Phi$ on this vector is:

$$\sigma_{\nu^o} U_{H_\Phi}^* \Gamma(h^*)(\Lambda_\Phi(g) \underset{\nu}{\beta} \otimes_\alpha (\lambda_{\Lambda_\Psi(\sigma_{-i}^{\Psi_i}(d^*))}^{\beta, \alpha})^* U_{H_\Psi}(\Lambda_\Psi(c) \underset{\nu^o}{\hat{\alpha}} \otimes_\beta \Lambda_\Phi((ab)^*)))$$

□

PROPOSITION 4.11. — *There exists a closed densely defined anti-linear operator G on H_Φ such that the linear span of:*

$$(\lambda_{\Lambda_\Psi(\sigma_{-i}^\Psi(b^*))}^{\beta,\alpha})^* U_{H_\Psi}(\Lambda_\Psi(a) \underset{\nu^o}{\alpha \otimes \hat{\beta}} \Lambda_\Phi((cd)^*))$$

with $a, c \in (\mathcal{N}_\Phi \cap \mathcal{N}_{T_L})^(\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})$, $b, d \in \mathcal{T}_{\Psi, T_R}$, is a core of G and we have:*

$$\begin{aligned} & G(\lambda_{\Lambda_\Psi(\sigma_{-i}^\Psi(b^*))}^{\beta,\alpha})^* U_{H_\Psi}(\Lambda_\Psi(a) \underset{\nu^o}{\alpha \otimes \hat{\beta}} \Lambda_\Phi((cd)^*)) \\ &= (\lambda_{\Lambda_\Psi(\sigma_{-i}^\Psi(d^*))}^{\beta,\alpha})^* U_{H_\Psi}(\Lambda_\Psi(c) \underset{\nu^o}{\alpha \otimes \hat{\beta}} \Lambda_\Phi((ab)^*)) \end{aligned}$$

Moreover, $G\mathcal{D}(G) = \mathcal{D}(G)$ and $G^2 = id|_{\mathcal{D}(G)}$.

Proof. — For all $n \in \mathbb{N}$, let $k_n \in \mathbb{N}$, $a(n, l), c(n, l) \in (\mathcal{N}_\Phi \cap \mathcal{N}_{T_L})^*(\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})$ and $b(n, l), d(n, l) \in \mathcal{T}_{\Psi, T_R}$ and let $w \in H_\Phi$ such that:

$$v_n = \sum_{l=1}^{k_n} (\lambda_{\Lambda_\Psi(\sigma_{-i}^\Psi(b(n, l)^*))}^{\beta,\alpha})^* U_{H_\Psi}(\Lambda_\Psi(a(n, l)) \underset{\nu^o}{\alpha \otimes \hat{\beta}} \Lambda_\Phi((c(n, l)d(n, l))^*)) \rightarrow 0$$

$$w_n = \sum_{l=1}^{k_n} (\lambda_{\Lambda_\Psi(\sigma_{-i}^\Psi(d(n, l)^*))}^{\beta,\alpha})^* U_{H_\Psi}(\Lambda_\Psi(c(n, l)) \underset{\nu^o}{\alpha \otimes \hat{\beta}} \Lambda_\Phi((a(n, l)b(n, l))^*)) \rightarrow w$$

We have $U_{H_\Phi}^* \Gamma(g^*)(\Lambda_\Phi(h) \underset{\nu}{\beta \otimes \alpha} v_n) \in \mathcal{D}(S_\Phi \underset{\nu^o}{\alpha \otimes \hat{\beta}} S_\Phi)$ for all $g, h \in \mathcal{T}_{\Phi, S_L}$ and $n \in \mathbb{N}$ by the previous proposition. Moreover, we have:

$$\sigma_\nu(S_\Phi \underset{\nu^o}{\alpha \otimes \hat{\beta}} S_\Phi) U_{H_\Phi}^* \Gamma(g^*)(\Lambda_\Phi(h) \underset{\nu}{\beta \otimes \alpha} v_n) = U_{H_\Phi}^* \Gamma(h^*)(\Lambda_\Phi(g) \underset{\nu}{\beta \otimes \alpha} w_n)$$

Since $\Lambda_\Phi(g)$ and $\Lambda_\Phi(h)$ belongs to $D((H_\Phi)_\beta, \nu^o)$, we obtain:

$$\sigma_\nu(S_\Phi \underset{\nu^o}{\alpha \otimes \hat{\beta}} S_\Phi) U_{H_\Phi}^* \Gamma(g^*) \lambda_{\Lambda_\Phi(h)}^{\beta,\alpha} v_n = U_{H_\Phi}^* \Gamma(h^*) \lambda_{\Lambda_\Phi(g)}^{\beta,\alpha} w_n$$

The closure of $S_\Phi \underset{\nu^o}{\alpha \otimes \hat{\beta}} S_\Phi$ implies that $U_{H_\Phi}^* \Gamma(h^*) \lambda_{\Lambda_\Phi(g)}^{\beta,\alpha} w = 0$. So, apply U_{H_Φ} ,

to get $\Gamma(h^*) \lambda_{\Lambda_\Phi(g)}^{\beta,\alpha} w = 0$. Now, \mathcal{T}_{Φ, S_L} is dense in M that's why $\lambda_{\Lambda_\Phi(g)}^{\beta,\alpha} w = 0$ for all $g \in \mathcal{T}_{\Phi, S_L}$. Then, by 3.8, we have:

$$\|\lambda_{\Lambda_\Phi(g)}^{\beta,\alpha} w\|^2 = (\alpha(\langle \Lambda_\Phi(g), \Lambda_\Phi(g) \rangle_{\beta, \nu^o}) w | w) = (S_L(\sigma_{i/2}^\Phi(g) \sigma_{-i/2}^\Phi(g^*)) w | w)$$

By density of \mathcal{T}_{Φ, S_L} , we obtain $\|w\|^2 = 0$ i.e $w = 0$. Consequently, the formula given in the proposition for G gives rise to a closable densely defined well-defined operator on H_Φ . So the required operator is the closure of the previous one. \square

Thanks to polar decomposition of the closed operator G , we can give the following definitions:

DEFINITION 4.12. — We denote by D the strictly positive operator G^*G on H_Φ (that means positive, self-adjoint and injective) and by I the anti-unitary operator on H_Φ such that $G = ID^{1/2}$.

Since G is involutive, we have $I = I^*$, $I^2 = 1$ and $IDI = D^{-1}$.

4.3. A fundamental commutation relation. — In this section, we establish a commutation relation between G and the elements $(\omega_{v,w} * id)(U'_{H_\Phi})$. We recall that $W' = U'_{H_\Psi}$. We begin by two lemmas borrowed from [Eno02].

LEMMA 4.13. — *Let ξ_i be a (N^o, ν^o) -basis of $(H_\Psi)_\beta$. For all $w' \in D(\hat{\alpha}H_\Psi, \nu)$ and $w \in H_\Psi$, we have:*

$$W'(w' \underset{\nu^o}{\hat{\alpha} \otimes \beta} w) = \sum_i \xi_i \underset{\nu}{\beta \otimes \alpha} (\omega_{w', \xi_i} * id)(W')w$$

*If we put $\delta_i = (\omega_{w', \xi_i} * id)(W')w$, then $\alpha(< \xi_i, \xi_i >_{\beta, \nu^o})\delta_i = \delta_i$. Moreover, if $w \in D(\hat{\alpha}(H_\Psi), \nu)$, then $\delta_i \in D(\hat{\alpha}(H_\Psi), \nu)$.*

For all $v, v' \in D((H_\Psi)_\beta, \nu^o)$ and $i \in I$, there exists $\zeta_i \in D((H_\Psi)_\beta, \nu^o)$ such that $\alpha(< \xi_i, \xi_i >_{\beta, \nu^o})\zeta_i = \zeta_i$ and:

$$W'(v' \underset{\nu^o}{\hat{\alpha} \otimes \beta} v) = \sum_i \xi_i \underset{\nu}{\beta \otimes \alpha} \zeta_i$$

Proof. — Lemma 3.4 of [Eno02]. □

REMARK 4.14. — If $v, v' \in \Lambda_\Psi(\mathcal{T}_{\Psi, T_R}) \subseteq D(\hat{\alpha}H, \nu) \cap D(H_\beta, \nu^o)$, then, with notations of the previous lemma, we have $\zeta_i \in D(\hat{\alpha}H, \nu) \cap D(H_\beta, \nu^o)$.

LEMMA 4.15. — *Let $v, v' \in D(H_\beta, \nu^o)$ and $w, w' \in D(\hat{\alpha}H, \nu)$. With notations of the previous lemma, we have:*

$$(\omega_{v,w} * id)(U_H^{t*})(\omega_{v',w'} * id)(U_H^{t*}) = \sum_i (\omega_{\zeta_i, \delta_i} * id)(U_H^{t*})$$

in the norm convergence (and also in the weak convergence).

Proof. — Proposition 3.6 of [Eno02]. □

LEMMA 4.16. — *Let a, c belonging to $(\mathcal{N}_\Phi \cap \mathcal{N}_{T_L})^*(\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})$. For all $b, d, a', b', c', d' \in \mathcal{T}_{\Psi, T_R}$, the value of $(\lambda_{\Lambda_\Psi(\sigma_{-i}^*(b^*))}^{\beta, \alpha})^* U_{H_\Psi}$ on the sum over i of:*

$$\Lambda_\Psi((\omega_{\Lambda_\Psi(ab), \xi_i} * id)(W')a') \underset{\nu^o}{\alpha \otimes \hat{\beta}} \Lambda_\Phi((c'd')^*(\omega_{\xi_i, \Lambda_\Psi(cd)} * id)(W'^*))$$

is equal to:

$$(\omega_{\Lambda_\Psi(a'b'), \Lambda_\Psi(c'd')} * id)(U_{H_\Phi}^{t*})(\lambda_{\Lambda_\Psi(\sigma_{-i}^*(b^*))}^{\beta, \alpha})^* U_{H_\Psi}(\Lambda_\Psi(a) \underset{\nu^o}{\alpha \otimes \hat{\beta}} \Lambda_\Phi((cd)^*))$$

Proof. — First, let's suppose that $a \in \mathcal{T}_{\Psi, T_R}$. By 3.17 and 3.19, we have:

$$\begin{aligned} & (\omega_{\Lambda_{\Psi}(a'b'), \Lambda_{\Psi}(c'd')} * id)(U_{H_{\Phi}}'^*) (\lambda_{\Lambda_{\Psi}(\sigma_{-i}^{\Psi}(b^*))}^{\beta, \alpha})^* U_{H_{\Psi}}(\Lambda_{\Psi}(a) \underset{\nu^o}{\alpha \otimes_{\beta}} \Lambda_{\Phi}((cd)^*)) \\ &= (\omega_{\Lambda_{\Psi}(a'b'), \Lambda_{\Psi}(c'd')} * id)(U_{H_{\Phi}}'^*) \Lambda_{\Phi}((\omega_{\Lambda_{\Psi}(a), \Lambda_{\Psi}(\sigma_{-i}^{\Psi}(b^*))} \underset{\nu}{\beta \otimes_{\alpha}} id)(\Gamma((cd)^*))) \\ &= (\omega_{\Lambda_{\Psi}(a'b'), \Lambda_{\Psi}(c'd')} * id)(U_{H_{\Phi}}'^*) \Lambda_{\Phi}((\omega_{\Lambda_{\Psi}(ab), \Lambda_{\Psi}(cd)} * id)(U_{H_{\Phi}}'^*)) \end{aligned}$$

By 4.15 and the closure of Λ_{Φ} , this expression is equal to the sum over $i \in I$ of:

$$\Lambda_{\Phi}((\omega_{\Lambda_{\Psi}(ab), \xi_i} * id)(W') \Lambda_{\Psi}(a'b'), (\omega_{\Lambda_{\Psi}(cd), \xi_i} * id)(W') \Lambda_{\Psi}(c'd') * id)(U_{H_{\Phi}}'^*))$$

Again, 3.17 and 3.19, we obtain the sum over $i \in I$ of the value of $(\lambda_{\Lambda_{\Psi}(\sigma_{-i}^{\Psi}(b'^*))}^{\beta, \alpha})^* U_{H_{\Psi}}$ on:

$$\Lambda_{\Psi}((\omega_{\Lambda_{\Psi}(ab), \xi_i} * id)(W') a') \underset{\nu^o}{\alpha \otimes_{\beta}} \Lambda_{\Phi}((c'd')^* (\omega_{\xi_i, \Lambda_{\Psi}(cd)} * id)(W'^*))$$

A density argument finishes the proof. \square

PROPOSITION 4.17. — If $v, w \in \Lambda_{\Psi}(\mathcal{T}_{\Psi, T_R}^2) \subseteq D(\hat{\alpha}(H_{\Psi}), \nu) \cap D((H_{\Psi})_{\beta}, \nu^o)$, then we have:

$$\begin{aligned} (4) \quad & (\omega_{v,w} * id)(U_{H_{\Phi}}'^*) G \subseteq G(\omega_{v,w} * id)(U_{H_{\Phi}}'^*) \\ (5) \quad & \text{and } (\omega_{v,w} * id)(U_{H_{\Phi}}'^*) G^* \subseteq G^*(\omega_{v,w} * id)(U_{H_{\Phi}}'^*) \end{aligned}$$

Proof. — Let $a, c \in (\mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L})^* (\mathcal{N}_{\Psi} \cap \mathcal{N}_{T_R})$ and $b, d, a', b', c', d' \in \mathcal{T}_{\Psi, T_R}$. By definition of G , we have:

$$(\lambda_{\Lambda_{\Psi}(\sigma_{-i}^{\Psi}(d^*))}^{\beta, \alpha})^* U_{H_{\Psi}}(\Lambda_{\Psi}(c) \underset{\nu^o}{\alpha \otimes_{\beta}} \Lambda_{\Phi}((ab)^*)) \in \mathcal{D}(G)$$

and:

$$\begin{aligned} & (\omega_{\Lambda_{\Psi}(a'b'), \Lambda_{\Psi}(c'd')} * id)(U_{H_{\Phi}}'^*) G(\lambda_{\Lambda_{\Psi}(\sigma_{-i}^{\Psi}(d^*))}^{\beta, \alpha})^* U_{H_{\Psi}}(\Lambda_{\Psi}(c) \underset{\nu^o}{\alpha \otimes_{\beta}} \Lambda_{\Phi}((ab)^*)) \\ &= (\omega_{\Lambda_{\Psi}(a'b'), \Lambda_{\Psi}(c'd')} * id)(U_{H_{\Phi}}'^*) (\lambda_{\Lambda_{\Psi}(\sigma_{-i}^{\Psi}(b^*))}^{\beta, \alpha})^* U_{H_{\Psi}}(\Lambda_{\Psi}(a) \underset{\nu^o}{\alpha \otimes_{\beta}} \Lambda_{\Phi}((cd)^*)) \end{aligned}$$

By the previous lemma, this is the sum over $i \in I$ of $G(\lambda_{\Lambda_{\Psi}(\sigma_{-i}^{\Psi}(d'^*))}^{\beta, \alpha})^* U_{H_{\Psi}}$ on:

$$\Lambda_{\Psi}((\omega_{\Lambda_{\Psi}(cd), \xi_i} * id)(W') c') \underset{\nu^o}{\alpha \otimes_{\beta}} \Lambda_{\Phi}((a'b')^* (\omega_{\xi_i, \Lambda_{\Psi}(ab)} * id)(W'^*))$$

Now, G is a closed operator, so we deduce that the sum over $i \in I$ of $(\lambda_{\Lambda_{\Psi}(\sigma_{-i}^{\Psi}(d'^*))}^{\beta, \alpha})^* U_{H_{\Psi}}$ on:

$$\Lambda_{\Psi}((\omega_{\Lambda_{\Psi}(cd), \xi_i} * id)(W') c') \underset{\nu^o}{\alpha \otimes_{\beta}} \Lambda_{\Phi}((a'b')^* (\omega_{\xi_i, \Lambda_{\Psi}(ab)} * id)(W'^*))$$

belongs to $\mathcal{D}(G)$ and by the previous lemma, we obtain:

$$\begin{aligned} & (\omega_{\Lambda_\Psi(a'b'), \Lambda_\Psi(c'd')} * id)(W'^*) G(\lambda_{\Lambda_\Psi(\sigma_{-i}^\Psi(d^*))}^{\beta, \alpha})^* U_{H_\Psi}(\Lambda_\Psi(c) \underset{\nu^o}{\alpha \otimes \beta} \Lambda_\Phi((ab)^*)) \\ &= G(\omega_{\Lambda_\Psi(c'd'), \Lambda_\Psi(a'b')} * id)(U_{H_\Phi}'^*)(\lambda_{\Lambda_\Psi(\sigma_{-i}^\Psi(d^*))}^{\beta, \alpha})^* U_{H_\Psi}(\Lambda_\Psi(c) \underset{\nu^o}{\alpha \otimes \beta} \Lambda_\Phi((ab)^*)) \end{aligned}$$

Now the linear span:

$$(\lambda_{\Lambda_\Psi(\sigma_{-i}^\Psi(b^*))}^{\beta, \alpha})^* U_{H_\Psi}(\Lambda_\Psi(a) \underset{\nu^o}{\alpha \otimes \beta} \Lambda_\Phi((cd)^*))$$

with $a, c \in (\mathcal{N}_\Phi \cap \mathcal{N}_{T_L})^*(\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})$, $b, d \in \mathcal{T}_{\Psi, T_R}\}$, is a core for G that's why the first inclusion holds. The second one is the adjoint of the first one. \square

COROLLARY 4.18. — *For all $v, w \in \Lambda_\Psi(\mathcal{T}_{\Psi, T_R}^2)$, we have:*

$$(\omega_{v,w} * id)(U_{H_\Phi}'^*) D \subseteq D(\omega_{\Delta_\Psi^{-1}v, \Delta_\Psi w} * id)(U_{H_\Phi}'^*)$$

where $D = G^*G$ is defined in 4.12.

Proof. — We have:

$$\begin{aligned} (\omega_{w,v} * id)(U_{H_\Phi}'^*) G &= (\omega_{S_\Psi w, \Delta_\Psi S_\Psi v} * id)(U_{H_\Phi}'^*) G && \text{by lemma 3.20} \\ &\subseteq G(\omega_{\Delta_\Psi S_\Psi v, S_\Psi w} \star id)(U_{H_\Phi}'^*) && \text{by inclusion (4)} \\ &= G(\omega_{\Delta_\Psi^{-1}v, \Delta_\Psi w} * id)(U_{H_\Phi}'^*) && \text{by lemma 3.20} \end{aligned}$$

In the same way, we can finish the proof:

$$\begin{aligned} (\omega_{v,w} * id)(U_{H_\Phi}'^*) D &= (\omega_{v,w} * id)(U_{H_\Phi}'^*) G^* && \text{by definition 4.12} \\ &\subseteq G^*(\omega_{w,v} * id)(U_{H_\Phi}'^*) G && \text{by inclusion (5)} \\ &\subseteq G^* G(\omega_{\Delta_\Psi^{-1}v, \Delta_\Psi w} * id)(U_{H_\Phi}'^*) \\ &= D(\omega_{\Delta_\Psi^{-1}v, \Delta_\Psi w} * id)(U_{H_\Phi}'^*) && \text{by definition 4.12.} \end{aligned}$$

\square

4.4. Scaling group. — In this section, we give a sense and we prove the following commutation relation $U_{H_\Phi}'^*(\Delta_\Psi \underset{\nu^o}{\hat{\alpha} \otimes \beta} D) = (\Delta_\Psi \underset{\nu}{\beta \otimes \alpha} D) U_{H_\Phi}'^*$ so as to construct the scaling group τ .

LEMMA 4.19. — *For all $\lambda \in \mathbb{C}$ and x analytic w.r.t ν , we have:*

$$\alpha(x) D^\lambda \subseteq D^\lambda \alpha(\sigma_{-i\lambda}^\nu(x)) \text{ and } \beta(x) D^\lambda \subseteq D^\lambda \beta(\sigma_{-i\lambda}^\nu(x))$$

Proof. — For all $a, c \in (\mathcal{N}_\Phi \cap \mathcal{N}_{T_L})^*(\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})$, $b, d \in \mathcal{T}_{\Psi, T_R}$ and x analytic w.r.t ν , we have by 3.21 and 3.22:

$$\begin{aligned}
& \beta(x)G(\lambda_{\Lambda_\Psi(\sigma_{-i}^\Psi(b^*))}^{\beta, \alpha})^*U_\Psi(\Lambda_\Psi(a) \underset{\nu^o}{\alpha \otimes_{\hat{\beta}}} \Lambda_\Phi((cd)^*)) \\
&= \beta(x)(\lambda_{\Lambda_\Psi(\sigma_{-i}^\Psi(d^*))}^{\beta, \alpha})^*U_\Psi(\Lambda_\Psi(c) \underset{\nu^o}{\alpha \otimes_{\hat{\beta}}} \Lambda_\Phi((ab)^*)) \\
&= (\lambda_{\Lambda_\Psi(\sigma_{-i}^\Psi(d^*))}^{\beta, \alpha})^*(1 \underset{\nu}{\beta \otimes \alpha} \beta(x))U_\Psi(\Lambda_\Psi(c) \underset{\nu^o}{\alpha \otimes_{\hat{\beta}}} \Lambda_\Phi((ab)^*)) \\
&= (\lambda_{\Lambda_\Psi(\sigma_{-i}^\Psi(d^*))}^{\beta, \alpha})^*U_\Psi(\Lambda_\Psi(c) \underset{\nu^o}{\alpha \otimes_{\hat{\beta}}} \Lambda_\Phi(\beta(x)b^*a^*)) \\
&= G(\lambda_{\Lambda_\Psi(\beta(\sigma_i^\nu(x))\sigma_{-i}^\Psi(b^*))}^{\beta, \alpha})^*U_\Psi(\Lambda_\Psi(a) \underset{\nu^o}{\alpha \otimes_{\hat{\beta}}} \Lambda_\Phi((cd)^*)) \\
&= G\alpha(\sigma_{-i/2}^\nu(x^*))(\lambda_{\Lambda_\Psi(\sigma_{-i}^\Psi(b^*))}^{\beta, \alpha})^*U_\Psi(\Lambda_\Psi(a) \underset{\nu^o}{\alpha \otimes_{\hat{\beta}}} \Lambda_\Phi((cd)^*))
\end{aligned}$$

Now, the linear span of:

$$(\lambda_{\Lambda_\Psi(\sigma_{-i}^\Psi(b^*))}^{\beta, \alpha})^*U_\Psi(\Lambda_\Psi(a) \underset{\nu^o}{\alpha \otimes_{\hat{\beta}}} \Lambda_\Phi((cd)^*))$$

where $a, c \in (\mathcal{N}_\Phi \cap \mathcal{N}_{T_L})^*(\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})$, $b, d \in \mathcal{T}_{\Psi, T_R}$, is a core for G , so that we have:

$$\beta(x)G \subseteq G\alpha(\sigma_{-i/2}^\nu(x^*))$$

Take adjoint to obtain $\alpha(x)G^* \subseteq G^*\beta(\sigma_{i/2}^\nu(x^*))$. So, we conclude by:

$$\alpha(x)D = \alpha(x)G^*G \subseteq G^*\beta(\sigma_{i/2}^\nu(x^*))G \subseteq D\alpha(\sigma_{-i}^\nu(x))$$

The second part of the lemma can be proved in a very similar way. \square

We now state two lemmas analogous to relations (2) and (3) for Ψ and we justify the existence of natural operators:

LEMMA 4.20. — For all $\lambda \in \mathbb{C}$, $x \in \mathcal{D}(\sigma_{-i\lambda}^\nu)$ and $\xi, \xi' \in \Lambda_\Psi(\mathcal{T}_{\Psi, T_R})$, we have:

$$\begin{aligned}
(6) \quad & \beta(x)\Delta_\Psi^\lambda \subseteq \Delta_\Psi^\lambda \beta(\sigma_{-i\lambda}^\nu(x)) \\
& R^{\beta, \nu^o}(\Delta_\Psi^\lambda \xi) \Delta_\nu^{-\lambda} \subseteq \Delta_\Psi^\lambda R^{\beta, \nu^o}(\xi) \\
& \text{and } \sigma_{-i\lambda}^\nu(< \Delta_\Psi^\lambda \xi, \xi' >_{\beta, \nu^o}) = < \xi, \Delta_\Psi^{\bar{\lambda}} \xi' >_{\beta, \nu^o}
\end{aligned}$$

and:

$$\begin{aligned}
(7) \quad & \hat{\alpha}(x)\Delta_\Psi^\lambda \subseteq \Delta_\Psi^\lambda \hat{\alpha}(\sigma_{-i\lambda}^\nu(x)) \\
& R^{\hat{\alpha}, \nu^o}(\Delta_\Psi^\lambda \xi) \Delta_\nu^{-\lambda} \subseteq \Delta_\Psi^\lambda R^{\hat{\alpha}, \nu^o}(\xi) \\
& \text{and } \sigma_{-i\lambda}^\nu(< \Delta_\Psi^\lambda \xi, \xi' >_{\hat{\alpha}, \nu^o}) = < \xi, \Delta_\Psi^{\bar{\lambda}} \xi' >_{\hat{\alpha}, \nu^o}
\end{aligned}$$

Proof. — It is sufficient to apply 4.5 to the opposite measured quantum groupoid for example. \square

LEMMA 4.21. — *We can define, for all $\lambda \in \mathbb{C}$, a closed linear operator $\Delta_{\Psi}^{\lambda} \beta \otimes_{\alpha} D^{\lambda}$ which naturally acts on elementary tensor products.*

Proof. — The proof is very similar to 4.6. \square

With relations (7) in hand, it's also possible to define a closed linear operator $\Delta_{\Psi}^{\lambda} \hat{\alpha} \otimes_{\beta} D^{\lambda}$ on $H_{\Psi} \hat{\alpha} \otimes_{\beta} H_{\Phi}$.

PROPOSITION 4.22. — *The following relation holds:*

$$(8) \quad U'_{H_{\Phi}}(\Delta_{\Psi} \hat{\alpha} \otimes_{\beta} D) = (\Delta_{\Psi} \beta \otimes_{\alpha} D)U'_{H_{\Phi}}$$

Proof. — By 4.18, we have, for all $v, w \in \Lambda_{\Psi}(\mathcal{T}_{\Psi, T_R})$ and $v', w' \in \mathcal{D}(D)$:

$$\begin{aligned} (U'_{H_{\Phi}}(v \hat{\alpha} \otimes_{\beta} v') | \Delta_{\Psi} w \beta \otimes_{\alpha} D w') &= ((\omega_{v, \Delta_{\Psi} w} * id)(U'_{H_{\Phi}} v') | D w') \\ &= (D(\omega_{\Delta_{\Psi}^{-1}(\Delta_{\Psi} v), \Delta_{\Psi} w} * id)(U'_{H_{\Phi}} v') | w') \\ &= ((\omega_{\Delta_{\Psi} v, w} * id)(U'_{H_{\Phi}}) D v' | w') \\ &= (U'_{H_{\Phi}}(\Delta_{\Psi} v \hat{\alpha} \otimes_{\beta} D v') | w \beta \otimes_{\alpha} w') \end{aligned}$$

By definition, we know that $\Lambda_{\Psi}(\mathcal{T}_{\Psi, T_R}) \odot \mathcal{D}(D)$ is a core for $\Delta_{\Psi} \beta \otimes_{\alpha} D$ so, for all $u \in \mathcal{D}(\Delta_{\Psi} \beta \otimes_{\alpha} D)$, we have:

$$(U'_{H_{\Phi}}(v \hat{\alpha} \otimes_{\beta} v') | (\Delta_{\Psi} \beta \otimes_{\alpha} D)u) = (U'_{H_{\Phi}}(\Delta_{\Psi} v \hat{\alpha} \otimes_{\beta} D v') | u)$$

Since $\Delta_{\Psi} \beta \otimes_{\alpha} D$ is self-adjoint, we get:

$$(\Delta_{\Psi} \beta \otimes_{\alpha} D)U'_{H_{\Phi}}(v \hat{\alpha} \otimes_{\beta} v') = U'_{H_{\Phi}}(\Delta_{\Psi} v \hat{\alpha} \otimes_{\beta} D v')$$

Finally, since $\Lambda_{\Psi}(\mathcal{T}_{\Psi, T_R}) \odot \mathcal{D}(D)$ is a core for $\Delta_{\Psi} \hat{\alpha} \otimes_{\beta} D$ and by closeness of $\Delta_{\Psi} \beta \otimes_{\alpha} D$, we deduce that:

$$U'_{H_{\Phi}}(\Delta_{\Psi} \hat{\alpha} \otimes_{\beta} D) \subseteq (\Delta_{\Psi} \beta \otimes_{\alpha} D)U'_{H_{\Phi}}$$

Because of unitarity of $U'_{H_{\Phi}}$, we get that $(\Delta_{\Psi} \hat{\alpha} \otimes_{\beta} D)U'^*_{H_{\Phi}} \subseteq U'^*_{H_{\Phi}}(\Delta_{\Psi} \beta \otimes_{\alpha} D)$ and by taking the adjoint, we get the reverse inclusion:

$$(\Delta_{\Psi} \beta \otimes_{\alpha} D)U'_{H_{\Phi}} \subseteq U'_{H_{\Phi}}(\Delta_{\Psi} \hat{\alpha} \otimes_{\beta} D)$$

\square

We now begin the construction of the scaling group τ strictly speaking. We also prove a theorem which states that $A(U'_H) = M$ and generalize proposition 1.5 of [KV03].

DEFINITION 4.23. — We denote by M_R the weakly closed linear span of:

$$\{(\omega \underset{\nu}{\beta} \star_{\alpha} id)(\Gamma(x)) \mid x \in M, \omega \in M_*^+ \text{ s.t } \exists k \in \mathbb{R}^+, \omega \circ \beta \leq k\nu\}$$

Also, we denote by M_L the weakly closed linear span of:

$$\{(id \underset{\nu}{\beta} \star_{\alpha} \omega)(\Gamma(x)) \mid x \in M, \omega \in M_*^+ \text{ s.t } \exists k \in \mathbb{R}^+, \omega \circ \alpha \leq k\nu\}$$

By 3.19 and 3.49, M_R is equal to the von Neumann subalgebra $A(U'_H)$ of M . Also, M_L is a von Neumann subalgebra of M . Moreover, we know $\alpha(N) \subseteq M_R$ and $\beta(N) \subseteq M_L$, so that $M_L \underset{\nu}{\beta} \star_{\alpha} M_R$ makes sense. Also, we have, for all $m \in M$:

$$(9) \quad \Gamma(m) \in M_L \underset{N}{\beta} \star_{\alpha} M_R$$

LEMMA 4.24. — *There exists a unique strongly continuous one-parameter group τ of automorphisms of M_R such that $\tau_t(x) = D^{-it}x D^{it}$ for all $t \in \mathbb{R}$ and $x \in M_R$.*

Proof. — By commutation relation (8), for all $t \in \mathbb{R}$ and $v, w \in \Lambda_{\Psi}(\mathcal{T}_{\Psi, T_R})$, we get that:

$$D^{-it}(\omega_{v,w} * id)(U'_{H_{\Phi}})D^{it} = (\omega_{\Delta_{\Psi}^{-it}v, \Delta_{\Psi}^{it}w} * id)(U'_{H_{\Phi}})$$

Consequently, we obtain $D^{-it}M_R D^{it} = M_R$ which is the only point to show. \square

LEMMA 4.25. — *We have $\tau_t(\alpha(n)) = \alpha(\sigma_t^{\nu}(n))$ for all $n \in N$ and $t \in \mathbb{R}$.*

Proof. — Straightforward by lemma 4.19. \square

LEMMA 4.26. — *It is possible to define a normal $*$ -automorphism $\sigma_t^{\Psi} \underset{N}{\beta} \star_{\alpha} \tau_{-t}$ of $M \underset{N}{\beta} \otimes_{\alpha} M_R$ which naturally acts for all $t \in \mathbb{R}$.*

Proof. — By the previous lemma and relations (6), we have, for all $n \in N$ and $t \in \mathbb{R}$:

$$\tau_t(\alpha(n)) = \alpha(\sigma_t^{\nu}(n)) \quad \text{and} \quad \sigma_t^{\Psi}(\beta(n)) = \beta(\sigma_{-t}^{\nu}(n))$$

so that it is possible to define a morphism:

$$\sigma_t^{\Psi} \underset{N}{\beta} \star_{\alpha} \tau_{-t} : M \underset{N}{\beta} \star_{\alpha} M_R \rightarrow M \underset{N}{\beta \circ \sigma_{-t}^{\nu}} \star_{\alpha \circ \sigma_{-t}^{\nu}} M_R$$

Then, it is sufficient to prove that the range is equal to the domain. For all $\xi \in D(H_\beta, \nu^o)$ and $y \in \mathcal{N}_\nu$, we compute:

$$\begin{aligned} \beta(\sigma_{-t}^\nu(y^*))\xi &= \beta(\sigma_{-t}^\nu(y)^*)\xi = R^{\beta, \nu^o}(\xi)J_\nu\Lambda_\nu(\sigma_{-t}^\nu(y)) \\ &= R^{\beta, \nu^o}(\xi)J_\nu\Delta_\nu^{-it}\Lambda_\nu(y) = R^{\beta, \nu^o}(\xi)\Delta_\nu^{-it}J_\nu\Lambda_\nu(y) \end{aligned}$$

and we get, for all $\xi \in D(H_\beta, \nu^o)$:

$$\xi \in D(H_{\beta \circ \sigma_{-t}^\nu}, \nu^o) \text{ and } R^{\beta \circ \sigma_{-t}^\nu, \nu^o}(\xi) = R^{\beta, \nu^o}(\xi)\Delta_\nu^{-it}$$

To conclude, we show that scalar products on $H \odot H$ used to define $H_{\beta \otimes_\alpha H}$ and $H_{\beta \circ \sigma_{-t}^\nu \otimes_{\alpha \circ \sigma_{-t}^\nu} H}$ are equal. For all $\xi, \xi' \in D(H_\beta, \nu^o)$ and $\eta, \eta' \in \check{H}_\nu$, we have:

$$\begin{aligned} (\xi_{\beta \circ \sigma_{-t}^\nu \otimes_{\alpha \circ \sigma_{-t}^\nu} \eta} | \xi'_{\beta \circ \sigma_{-t}^\nu \otimes_{\alpha \circ \sigma_{-t}^\nu} \eta'}) &= (\alpha(\sigma_{-t}^\nu(< \xi, \xi' >_{\beta \circ \sigma_{-t}^\nu, \nu^o}))\eta | \eta') \\ &= (\alpha(\sigma_{-t}^\nu(\Delta_\nu^{it} < \xi, \xi' >_{\beta, \nu^o} \Delta_\nu^{-it}))\eta | \eta') \\ &= (\alpha(< \xi, \xi' >_{\beta, \nu^o})\eta | \eta') \\ &= (\xi_{\beta \otimes_\alpha \eta} | \xi'_{\beta \otimes_\alpha \eta'}) \end{aligned}$$

□

PROPOSITION 4.27. — We have $(\sigma_t^\Psi \beta \star_\alpha \tau_{-t}) \circ \Gamma = \Gamma \circ \sigma_t^\Psi$ for all $t \in \mathbb{R}$.

Proof. — By relation (9), the formula makes sense (τ is just defined on M_R). By relation (8), we can compute for all $m \in M$ and $t \in \mathbb{R}$:

$$\begin{aligned} (\sigma_t^\Psi \beta \star_\alpha \tau_{-t}) \circ \Gamma(m) &= (\Delta_\Psi^{it} \beta \otimes_\alpha D^{it})\Gamma(m)(\Delta_\Psi^{-it} \beta \otimes_\alpha D^{-it}) \\ &= (\Delta_\Psi^{it} \beta \otimes_\alpha D^{it})U'_{H_\Phi}(m_{\hat{\alpha} \otimes_{\beta} 1})U_{H_\Phi}^*(\Delta_\Psi^{-it} \beta \otimes_\alpha D^{-it}) \\ &= U'_{H_\Phi}(\Delta_\Psi^{it} \hat{\alpha} \otimes_\beta D^{it})(m_{\hat{\alpha} \otimes_{\beta} 1})(\Delta_\Psi^{-it} \hat{\alpha} \otimes_\beta D^{-it})U_{H_\Phi}^* \\ &= U'_{H_\Phi}(\sigma_t^\Psi(m)_{\hat{\alpha} \otimes_\beta 1})U_{H_\Phi}^* = \Gamma(\sigma_t^\Psi(m)) \end{aligned}$$

□

We are now able to prove that we can re-construct M thanks to the fundamental unitary.

THEOREM 4.28. — *If $\langle F \rangle^{-w}$ is the weakly closed linear span of F in M , then following vector spaces:*

$$M_R = \langle (\omega \underset{\nu}{\beta} \star_{\alpha} id)(\Gamma(m)) \mid m \in M, \omega \in M_*^+, k \in \mathbb{R}^+ \text{ s.t } \omega \circ \beta \leq k\nu \rangle^{-w}$$

$$A(U'_H) = \langle (\omega_{v,w} * id)(U'_H) \mid v \in D(\hat{\alpha}(H_{\Psi}), \mu), w \in D((H_{\Psi})_{\beta}, \mu^o) \rangle^{-w}$$

$$M_L = \langle (id \underset{\nu}{\beta} \star_{\alpha} \omega)(\Gamma(m)) \mid m \in M, \omega \in M_*^+, k \in \mathbb{R}^+ \text{ s.t } \omega \circ \alpha \leq k\nu \rangle^{-w}$$

$$A(U_H) = \langle (id * \omega_{v,w})(U_H) \mid v \in D((H_{\Psi}), \mu^o)_{\hat{\beta}}, w \in D(\alpha(H_{\Psi}), \mu) \rangle^{-w}$$

are equal to the whole von Neumann algebra M .

Proof. — We have already noticed that $M_R = A(U'_H)$ and $M_L = A(U_H)$. Then, we get inspired by [KV03]. By 4.25, we have $\tau_t(\alpha(n)) = \alpha(\sigma_t^{\nu}(n))$ so:

$$M_L = \langle (id \underset{\nu}{\beta} \star_{\alpha} \omega \circ \tau_t)(\Gamma(m)) \mid m \in M, \omega \in (M_R)_*^+, k \in \mathbb{R}^+ \text{ s.t } \omega \circ \alpha \leq k\nu \rangle^{-w}$$

By 4.27, we have $\sigma_t^{\Psi}((id \underset{\nu}{\beta} \star_{\alpha} \omega)(\Gamma(m))) = (id \underset{\nu}{\beta} \star_{\alpha} \omega \circ \tau_t)(\Gamma(\sigma_t^{\Psi}(m)))$ that's why $\sigma_t^{\Psi}(M_L) = M_L$ for all $t \in \mathbb{R}$. On the other hand, by 3.11, restriction of Ψ to M_L is semi-finite. By Takesaki's theorem ([Str81], theorem 10.1), there exists a unique normal and faithful conditional expectation E from M to M_L such that $\Psi(m) = \Psi(E(m))$ for all $m \in M^+$. Moreover, if P is the orthogonal projection on the closure of $\Lambda_{\Psi}(\mathcal{N}_{\Psi} \cap M_L)$ then $E(m)P = PmP$.

So the range of P contains $\Lambda_{\Psi}((id \underset{\nu}{\beta} \star_{\alpha} \omega)(\Gamma(x)))$ for all ω and $x \in \mathcal{N}_{\Psi}$. By right version of 3.37 implies that $P = 1$ so that E is the identity and $M = M_L$. If we apply the previous result to the opposite measured quantum groupoid, then we get that $M = M_R$. \square

COROLLARY 4.29. — *There exists a unique strongly continuous one-parameter group τ of automorphisms of M such that, for all $t \in \mathbb{R}$, $m \in M$ and $n \in N$:*

$$\tau_t(m) = D^{-it} m D^{it}, \quad \tau_t(\alpha(n)) = \alpha(\sigma_t^{\nu}(n)) \text{ and } \tau_t(\beta(n)) = \beta(\sigma_t^{\nu}(n))$$

Proof. — Straightforward from the previous theorem and 4.24. First property comes from 4.25 and the second one from 4.19. \square

DEFINITION 4.30. — The one-parameter group τ is called **scaling group**.

LEMMA 4.31. — *It is possible to define normal $*$ -automorphisms $\tau_t \underset{N}{\beta} \star_{\alpha} \tau_t$ and $\tau_t \underset{N}{\beta} \star_{\alpha} \sigma_t^{\Phi}$ of $M \underset{N}{\beta} \otimes_{\alpha} M$ for all $t \in \mathbb{R}$.*

Proof. — The proof is very similar to 4.26. \square

PROPOSITION 4.32. — *We have $\Gamma \circ \tau_t = (\tau_t \underset{N}{\beta} \star_{\alpha} \tau_t) \circ \Gamma$ for all $t \in \mathbb{R}$.*

Proof. — By 4.27 and co-product relation, we have for all $t \in \mathbb{R}$:

$$\begin{aligned}
(id \underset{\nu}{\beta \star_{\alpha}} \Gamma)(\sigma_t^{\Psi} \underset{\nu}{\beta \star_{\alpha}} \tau_{-t}) \circ \Gamma &= (id \underset{\nu}{\beta \star_{\alpha}} \Gamma) \Gamma \circ \sigma_t^{\Psi} \\
&= (\Gamma \underset{\nu}{\beta \star_{\alpha}} id) \Gamma \circ \sigma_t^{\Psi} = (\Gamma \circ \sigma_t^{\Psi} \underset{\nu}{\beta \star_{\alpha}} \tau_{-t}) \Gamma \\
&= (\sigma_t^{\Psi} \underset{\nu}{\beta \star_{\alpha}} \tau_{-t} \underset{\nu}{\beta \star_{\alpha}} \tau_{-t})(\Gamma \underset{\nu}{\beta \star_{\alpha}} id) \Gamma \\
&= (\sigma_t^{\Psi} \underset{\nu}{\beta \star_{\alpha}} [(\tau_{-t} \underset{\nu}{\beta \star_{\alpha}} \tau_{-t}) \circ \Gamma]) \circ \Gamma
\end{aligned}$$

Consequently, for all $m \in M$, $\omega \in M_*^+$, $k \in \mathbb{R}^+$ such that $\omega \circ \beta \leq k\nu$, we have:

$$\begin{aligned}
\Gamma \circ \tau_{-t} \circ ((\omega \circ \sigma_t^{\Psi}) \underset{\nu}{\beta \star_{\alpha}} id) \Gamma &= (\omega \underset{\nu}{\beta \star_{\alpha}} id \underset{\nu}{\beta \star_{\alpha}} id)(\sigma_t^{\Psi} \underset{\nu}{\beta \star_{\alpha}} (\Gamma \circ \tau_{-t})) \circ \Gamma \\
&= (\omega \underset{\nu}{\beta \star_{\alpha}} id \underset{\nu}{\beta \star_{\alpha}} id)(\sigma_t^{\Psi} \underset{\nu}{\beta \star_{\alpha}} [(\tau_{-t} \underset{\nu}{\beta \star_{\alpha}} \tau_{-t}) \circ \Gamma]) \\
&= [(\tau_{-t} \underset{\nu}{\beta \star_{\alpha}} \tau_{-t}) \circ \Gamma] \circ ((\omega \circ \sigma_t^{\Psi}) \underset{\nu}{\beta \star_{\alpha}} id) \Gamma
\end{aligned}$$

The theorem 4.28 allows us to conclude. \square

PROPOSITION 4.33. — *For all $x \in M \cap \alpha(N)'$, we have $\Gamma(x) = 1 \underset{N}{\beta \otimes_{\alpha}} x \Leftrightarrow x \in \beta(N)$. Also, for all $x \in M \cap \beta(N)'$, we have $\Gamma(x) = x \underset{N}{\beta \otimes_{\alpha}} 1 \Leftrightarrow x \in \alpha(N)$.*

Proof. — Let $x \in M \cap \alpha(N)'$ such that $\Gamma(x) = 1 \underset{N}{\beta \otimes_{\alpha}} x$. For all $n \in \mathbb{N}$, we define in the strong topology:

$$x_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2 t^2) \sigma_t^{\Psi}(x) dt \quad \text{analytic w.r.t } \sigma^{\Psi},$$

and:

$$y_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2 t^2) \tau_{-t}(x) dt \quad \text{belongs to } \alpha(N)'.$$

By 4.27, we have $\Gamma(x_n) = 1 \underset{N}{\beta \otimes_{\alpha}} y_n$. If $d \in (\mathcal{M}_{\Psi} \cap \mathcal{M}_{T_R})^+$, then, for all $n \in \mathbb{N}$, we have $dx_n \in \mathcal{M}_{\Psi} \cap \mathcal{M}_{T_R}$. Let $\omega \in M_*^+$ and $k \in \mathbb{R}^+$ such that $\omega \circ \alpha \leq k\nu$. By right invariance, we get:

$$\begin{aligned}
\omega \circ T_R(dx_n) &= \omega((\Psi \underset{\nu}{\beta \star_{\alpha}} id)(\Gamma(dx_n))) \\
&= \Psi((id \underset{\nu}{\beta \star_{\alpha}} \omega)(\Gamma(dx_n))) = \Psi((id \underset{\nu}{\beta \star_{\alpha}} (y_n \omega))(\Gamma(d))) \\
&= \omega((\Psi \underset{\nu}{\beta \star_{\alpha}} id)(\Gamma(d))y_n) = \omega(T_R(d)y_n)
\end{aligned}$$

Take the limit over $n \in \mathbb{N}$ to obtain $T_R(dx) = T_R(d)x$ for all $d \in \mathcal{M}_{\Psi} \cap \mathcal{M}_{T_R}$ and, by semi-finiteness of T_R , we conclude that x belongs to $\beta(N)$. Reverse

inclusion comes from axioms. If we apply this result to the opposite measured quantum groupoid, then we get the second point. \square

4.5. The antipode and its polar decomposition. — We now approach definition of the antipode.

LEMMA 4.34. — *We have $(\omega_{v,w} * id)(U'_{H_\Phi})D^\lambda \subset D^\lambda(\omega_{\Delta_\Psi^{-\lambda}v, \Delta_\Psi^\lambda w} * id)(U'_{H_\Phi})$ for all $\lambda \in \mathbb{C}$ and $v, w \in \Lambda_\Psi(\mathcal{T}_\Psi, T_R)$.*

Proof. — Straightforward from relation (8). \square

PROPOSITION 4.35. — *If I is the unitary part of the polar decomposition of G , then, for all $v, w \in D((H_\Psi)_\beta, \nu^o)$, we have:*

$$I(\omega_{J_\Psi w, v} * id)(U'^*_{H_\Phi})I = (\omega_{J_\Psi v, w} * id)(U'_{H_\Phi})$$

Proof. — We have $(\omega_{v,w} * id)(U'_{H_\Phi})D^{1/2} \subseteq D^{1/2}(\omega_{\Delta_\Psi^{-1/2}v, \Delta_\Psi^{1/2}w} * id)(U'_{H_\Phi})$ for all $v, w \in \Lambda_\Psi(\mathcal{T}_\Psi, T_R)$ by the previous lemma. On the other hand, by inclusion (5), we have:

$$(\omega_{v,w} * id)(U'_{H_\Phi})D^{1/2} = (\omega_{v,w} * id)(U'_{H_\Phi})G^*I \subseteq D^{1/2}I(\omega_{w,v} * id)(U'_{H_\Phi})I$$

So $I(\omega_{w,v} * id)(U'_{H_\Phi})I = (\omega_{\Delta_\Psi^{-1/2}v, \Delta_\Psi^{1/2}w} * id)(U'_{H_\Phi})$ and, by 3.20, we have:

$$I(\omega_{w,v} * id)(U'^*_{H_\Phi})I = (\omega_{\Delta_\Psi^{1/2}w, \Delta_\Psi^{-1/2}v} * id)(U'^*_{H_\Phi}) = (\omega_{J_\Psi v, J_\Psi w} * id)(U'_{H_\Phi})$$

\square

COROLLARY 4.36. — *There exists a $*$ -anti-automorphism R of M defined by $R(m) = Im^*I$ such that $R^2 = id$. (We recall that I denotes the unitary part of the polar decomposition of G).*

Proof. — Straightforward from the previous proposition and theorem 4.28. \square

DEFINITION 4.37. — The unique $*$ -anti-automorphism R of M such that $R(m) = Im^*I$, where I denotes the unitary part of the polar decomposition of G , is called **unitary antipode**.

DEFINITION 4.38. — The application $S = R\tau_{-i/2}$ is called **antipode**.

The next proposition states elementary properties of the antipode. Straightforward proofs are omitted.

PROPOSITION 4.39. — *The antipode S satisfies:*

- i) *for all $t \in \mathbb{R}$, we have $\tau_t \circ R = R \circ \tau_t$ and $\tau_t \circ S = S \circ \tau_t$*
- ii) *$SR = RS$ and $S^2 = \tau_{-i}$*
- iii) *S is densely defined and has dense range*
- iv) *S is injective and $S^{-1} = R\tau_{i/2} = \tau_{i/2}R$*
- v) *for all $x \in \mathcal{D}(S)$, $S(x^*) \in \mathcal{D}(S)$ and $S(S(x)^*)^* = x$*

4.6. Characterization of the antipode. — In 4.38, we define the antipode by giving its polar decomposition. However, we have to verify that S is what it should be.

4.6.1. *Usual characterization of the antipode.* —

PROPOSITION 4.40. — *For all $v, w \in \Lambda_\Psi(\mathcal{T}_\Psi, T_R)$, $(\omega_{w,v} * id)(U'_{H_\Phi})$ belongs to $\mathcal{D}(S)$ and we have:*

$$S((\omega_{w,v} * id)(U'_{H_\Phi})) = (\omega_{w,v} * id)(U'^*_{H_\Phi})$$

Moreover, the linear span of $(\omega_{v,w} * id)(U'_{H_\Phi})$, where $v, w \in \Lambda_\Psi(\mathcal{T}_\Psi, T_R)$, is a core for S .

Proof. — By 4.34, we have $(\omega_{w,v} * id)(U'_{H_\Phi}) \in \mathcal{D}(\tau_{-i/2}) = \mathcal{D}(S)$ and:

$$\begin{aligned} S((\omega_{w,v} * id)(U'_{H_\Phi})) &= R((\omega_{\Delta_\Psi^{-1/2}w, \Delta_\Psi^{1/2}v} * id)(U'_{H_\Phi})) \\ &= (\omega_{S_\Psi v, \Delta_\Psi S_\Psi w} * id)(U'_{H_\Phi}) && \text{by proposition 4.35,} \\ &= (\omega_{w,v} * id)(U'^*_{H_\Phi}) && \text{by lemma 3.20.} \end{aligned}$$

The involved subspace of M is included in $\mathcal{D}(\tau_{-i/2})$ by 4.34, weakly dense in M by theorem 4.28 and τ -invariant by 4.24 which finishes the proof. \square

COROLLARY 4.41. — *For $a, b, c, d \in \mathcal{T}_\Psi, T_R$, $(\omega_{\Lambda_\Psi(a), \Lambda_\Psi(b)} \beta_\nu^{\star\alpha} id)(\Gamma(cd))$ belongs to $\mathcal{D}(S)$ and we have:*

$$S((\omega_{\Lambda_\Psi(a), \Lambda_\Psi(b)} \beta_\nu^{\star\alpha} id)(\Gamma(cd))) = (\omega_{\Lambda_\Psi(c), \Lambda_\Psi(\sigma_{-i}^\Psi(d^*))} \beta_\nu^{\star\alpha} id)(\Gamma(\sigma_i^\Psi(a)b^*))$$

Proof. — By 3.19, we know that:

$$(\omega_{\Lambda_\Psi(a), \Lambda_\Psi(b)} \beta_\nu^{\star\alpha} id)(\Gamma(cd)) = (\omega_{\Lambda_\Psi(cd), \Lambda_\Psi(b\sigma_{-i}^\Psi(a^*))} * id)(U'_{H_\Phi})$$

which belongs to $\mathcal{D}(S)$. Then, by 3.19 and 3.20, we have:

$$\begin{aligned} S((\omega_{\Lambda_\Psi(a), \Lambda_\Psi(b)} \beta_\nu^{\star\alpha} id)(\Gamma(cd))) &= S((\omega_{\Lambda_\Psi(cd), \Lambda_\Psi(b\sigma_{-i}^\Psi(a^*))} * id)(W')) \\ &= (\omega_{\Lambda_\Psi(cd), \Lambda_\Psi(b\sigma_{-i}^\Psi(a^*))} * id)(W'^*) \\ &= (\omega_{\Lambda_\Psi(\sigma_i^\Psi(a)b^*), \Lambda_\Psi(\sigma_{-i}^\Psi(d^*c^*))} * id)(W') \\ &= (\omega_{\Lambda_\Psi(c), \Lambda_\Psi(\sigma_{-i}^\Psi(d^*))} \beta_\nu^{\star\alpha} id)(\Gamma(\sigma_i^\Psi(a)b^*)) \end{aligned}$$

\square

4.6.2. *The co-involution R .* — In this section, we give a new expression of R and we show that it is a co-involution of the measured quantum groupoid.

PROPOSITION 4.42. — *For all $a, b \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$, we have:*

$$R((\omega_{J_\Psi \Lambda_\Psi(a)} \underset{\nu}{\beta \star_\alpha} id)(\Gamma(b^*b))) = (\omega_{J_\Psi \Lambda_\Psi(b)} \underset{\nu}{\beta \star_\alpha} id)(\Gamma(a^*a))$$

Proof. — The proposition comes from the following computation:

$$\begin{aligned} & R((\omega_{J_\Psi \Lambda_\Psi(a), J_\Psi \Lambda_\Psi(a)} \underset{\nu}{\beta \star_\alpha} id)(\Gamma(b^*b))) \\ &= R((\omega_{\Lambda_\Psi(b^*b), J_\Psi \Lambda_\Psi(a^*a)} * id)(U'_{H_\Phi})) && \text{by corollary 3.19,} \\ &= (\omega_{\Lambda_\Psi(a^*a), J_\Psi \Lambda_\Psi(b^*b)} * id)(U'_{H_\Phi}) && \text{by definition of } R, \\ &= (\omega_{J_\Psi \Lambda_\Psi(b), J_\Psi \Lambda_\Psi(b)} \underset{\nu}{\beta \star_\alpha} id)(\Gamma(a^*a)) && \text{by corollary 3.19.} \end{aligned}$$

□

REMARK 4.43. — We notice that R is T_L -independent.

PROPOSITION 4.44. — *We have $I\alpha(n^*) = \beta(n)I$ for all $n \in N$ and $R \circ \alpha = \beta$.*

Proof. — By 4.19, we have, for all $x \in \mathcal{T}_{\Psi, T_R}$:

$$\beta(x)GD^{-1/2} \subseteq G\alpha(\sigma_{-i/2}((x^*))) \subseteq GD^{-1/2}\alpha(x^*) \subseteq I\alpha(x^*)$$

and, on the other hand, $\beta(x)GD^{-1/2} \subseteq \beta(x)I$ so that $I\alpha(x^*) = \beta(x)I$. The result holds by normality of α and β . □

By [Sau83b], there exists a unitary and anti-linear operator $I \underset{\nu}{\beta \otimes_\alpha} I$ from $H \underset{\nu}{\beta \otimes_\alpha} H$ onto $H \underset{\nu^\circ}{\alpha \otimes_\beta} H$, the adjoint of which is $I \underset{\nu^\circ}{\alpha \otimes_\beta} I$. Also, there exists an anti-isomorphism $R \underset{N}{\beta \star_\alpha} R$ from $M \underset{N}{\beta \star_\alpha} M$ onto $M \underset{N^\circ}{\alpha \star_\beta} M$ and, by definition of R , we have, for all $X \in M \underset{N}{\beta \star_\alpha} M$:

$$(R \underset{N}{\beta \star_\alpha} R)(X) = (I \underset{\nu}{\beta \otimes_\alpha} I)X^*(I \underset{\nu^\circ}{\alpha \otimes_\beta} I)$$

We underline the fact that, if $\omega \in M_*^+$, then $\omega \circ R \in M_*^+$ and, if there exists $k \in \mathbb{R}^+$ such that $\omega \circ \alpha \leq k\nu$, then $\omega \circ R \circ \beta \leq k\nu$. Also, if $\theta \in M_*^+$ and $k' \in \mathbb{R}^+$ are such that $\theta \circ \beta \leq k'\nu$, then $\theta \circ R \circ \alpha \leq k'\nu$. Then, we have $\omega R \underset{\nu}{\beta \star_\alpha} \theta R = (\omega \underset{\nu^\circ}{\alpha \star_\beta} \theta) \circ (R \underset{\nu}{\beta \star_\alpha} R)$.

LEMMA 4.45. — *For all $a, x \in \mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$, $\omega \in M_*^+$ and $k \in \mathbb{R}^+$ such that $\omega \circ \alpha \leq k\nu$, we have:*

$$\omega \circ R((\omega_{J_\Psi \Lambda_\Psi(a)} \underset{\nu}{\beta \star_\alpha} id)(\Gamma(x))) = (\Lambda_\Psi((id \underset{\nu}{\beta \star_\alpha} \omega)(\Gamma(a^*a)))|J_\Psi \Lambda_\Psi(x))$$

Proof. — Let $b \in \mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$. By 4.42, we can compute:

$$\begin{aligned}
 \omega \circ R((\omega_{J_\Psi \Lambda_\Psi(a)} \underset{\nu}{\beta \star_\alpha} id)(\Gamma(b^*b))) &= \omega((\omega_{J_\Psi \Lambda_\Psi(b)} \underset{\nu}{\beta \star_\alpha} id)(\Gamma(a^*a))) \\
 &= ((id \underset{\nu}{\beta \star_\alpha} \omega)(\Gamma(a^*a))J_\Psi \Lambda_\Psi(b)|J_\Psi \Lambda_\Psi(b)) \\
 &= (J_\Psi b J_\Psi \Lambda_\Psi((id \underset{\nu}{\beta \star_\alpha} \omega)(\Gamma(a^*a))|J_\Psi \Lambda_\Psi(b)) \\
 &= (\Lambda_\Psi((id \underset{\nu}{\beta \star_\alpha} \omega)(\Gamma(a^*a))|J_\Psi \Lambda_\Psi(b^*b))
 \end{aligned}$$

Linearity and normality of the expressions imply the lemma. \square

PROPOSITION 4.46. — We have $\varsigma_{N^\circ} \circ (R \underset{N}{\beta \star_\alpha} R) \circ \Gamma = \Gamma \circ R$.

Proof. — Let $a, b \in \mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$, $\omega, \theta \in M_*^+$ and $k, k' \in \mathbb{R}^+$ such that $\omega \circ \alpha \leq k\nu$ and $\theta \circ \beta \leq k'\nu$. Then, we can compute by 4.42 and the previous lemma:

$$\begin{aligned}
 &(\theta \underset{\nu}{\beta \star_\alpha} \omega)(\Gamma \circ R((\omega_{J_\Psi \Lambda_\Psi(a)} \underset{\nu}{\beta \star_\alpha} id)(\Gamma(b^*b)))) \\
 &= (\theta \underset{\nu}{\beta \star_\alpha} \omega)(\Gamma((\omega_{J_\Psi \Lambda_\Psi(b)} \underset{\nu}{\beta \star_\alpha} id)(\Gamma(a^*a)))) \\
 &= (\omega_{J_\Psi \Lambda_\Psi(b)} \underset{\nu}{\beta \star_\alpha} \theta \underset{\nu}{\beta \star_\alpha} \omega)(id \underset{N}{\beta \star_\alpha} \Gamma)(\Gamma(a^*a)) \\
 &= (\omega_{J_\Psi \Lambda_\Psi(b)} \underset{\nu}{\beta \star_\alpha} \theta \underset{\nu}{\beta \star_\alpha} \omega)(\Gamma \underset{N}{\beta \star_\alpha} id)(\Gamma(a^*a)) \\
 &= (\omega_{J_\Psi \Lambda_\Psi(b)} \underset{\nu}{\beta \star_\alpha} \theta)[\Gamma((id \underset{\nu}{\beta \star_\alpha} \omega)(\Gamma(a^*a)))] \\
 &= (\Lambda_\Psi((id \underset{\nu}{\beta \star_\alpha} \theta \circ R)(\Gamma(b^*b))|J_\Psi \Lambda_\Psi((id \underset{\nu}{\beta \star_\alpha} \omega)(\Gamma(a^*a))))
 \end{aligned}$$

Observe the symmetry of the last expression and use it to proceed towards the computation:

$$\begin{aligned}
 &(\Lambda_\Psi((id \underset{\nu}{\beta \star_\alpha} \omega)(\Gamma(a^*a))|J_\Psi \Lambda_\Psi((id \underset{\nu}{\beta \star_\alpha} \theta \circ R)(\Gamma(b^*b))))) \\
 &= (\omega_{J_\Psi \Lambda_\Psi(a)} \underset{\nu}{\beta \star_\alpha} \omega \circ R)[\Gamma((id \underset{\nu}{\beta \star_\alpha} \theta \circ R)(\Gamma(b^*b)))] \\
 &= (\omega_{J_\Psi \Lambda_\Psi(a)} \underset{\nu}{\beta \star_\alpha} \omega \circ R \underset{\nu}{\beta \star_\alpha} \theta \circ R)(\Gamma \underset{N}{\beta \star_\alpha} id)(\Gamma(b^*b)) \\
 &= (\omega_{J_\Psi \Lambda_\Psi(a)} \underset{\nu}{\beta \star_\alpha} \omega \circ R \underset{\nu}{\beta \star_\alpha} \theta \circ R)(id \underset{N}{\beta \star_\alpha} \Gamma)(\Gamma(b^*b)) \\
 &= (\omega \circ R \underset{\nu}{\beta \star_\alpha} \theta \circ R)(\Gamma((\omega_{J_\Psi \Lambda_\Psi(a)} \underset{\nu}{\beta \star_\alpha} id)(\Gamma(b^*b)))) \\
 &= (\omega \underset{\nu^\circ}{\alpha \star_\beta} \theta)(R \underset{N}{\beta \star_\alpha} R)(\Gamma((\omega_{J_\Psi \Lambda_\Psi(a)} \underset{\nu}{\beta \star_\alpha} id)(\Gamma(b^*b)))) \\
 &= (\theta \underset{\nu}{\beta \star_\alpha} \omega) \varsigma_{N^\circ}(R \underset{N}{\beta \star_\alpha} R)(\Gamma((\omega_{J_\Psi \Lambda_\Psi(a)} \underset{\nu}{\beta \star_\alpha} id)(\Gamma(b^*b))))
 \end{aligned}$$

Theorem 4.28 easily implies the result. \square

4.6.3. *Left strong invariance w.r.t the antipode.*— In this section, T' denotes a left invariant n.s.f weight from M to $\alpha(N)$. We put $\Phi' = \nu \circ \alpha^{-1} \circ T'$, $J_{\Phi'}$ the anti-linear operator and $\Delta_{\Phi'}$ the modular operator which come from Tomita's theory of Φ' , $\sigma^{\Phi'}$ its modular group and $V = (U_{T'})_{H_{\Phi}}^*$ i.e the fundamental unitary associated with T' . The next proposition is the left strong invariance w.r.t S .

PROPOSITION 4.47. — *Elements $(id * \omega_{v,w})(V)$ belong to the domain of S for all $v, w \in \Lambda_{\Phi'}(\mathcal{T}_{\Phi', T'})$ and we have $S((id * \omega_{v,w})(V)) = (id * \omega_{v,w})(V^*)$.*

Proof. — By 3.19, we have $(id * \omega)(V) = (\omega \circ R * id)(U'_{H_{\Phi}})$ for all ω . If $\overline{\omega}(x) = \overline{\omega(x^*)}$, then, by 4.40, we have:

$$\begin{aligned} S((id * \omega)(V)) &= S((\omega \circ R * id)(U'_{H_{\Phi}})) = (\omega \circ R * id)(U'^*_{H_{\Phi}}) \\ &= [(\overline{\omega} \circ R * id)(U'_{H_{\Phi}})]^* \\ &= [(id * \overline{\omega})(V)]^* = (id * \omega)(V^*) \end{aligned}$$

□

LEMMA 4.48. — *For all $v \in \mathcal{D}(D^{1/2})$ and $w \in \mathcal{D}(D^{1/2})$, we have:*

$$(\omega_{v,w} * id)(V)^* = (\omega_{ID^{-1/2}v, ID^{1/2}w} * id)(V)$$

Proof. — We have $(id * \omega_{w',v'})(V) \in \mathcal{D}(S) = \mathcal{D}(\tau_{-i/2})$ for all v', w' belonging to $\Lambda_{\Phi'}(\mathcal{T}_{\Phi', T'})$ by 4.47 and, since τ is implemented by D^{-1} , we have:

$$\begin{aligned} (id * \omega_{w',v'})(V)D^{1/2} &\subseteq D^{1/2}\tau_{-i/2}((id * \omega_{w',v'})(V)) \\ &= D^{1/2}R(S((id * \omega_{w',v'})(V))) \\ &= D^{1/2}I[(id * \omega_{w',v'})(V^*)]^*I \\ &= D^{1/2}I(id * \omega_{v',w'})(V)I. \end{aligned}$$

Then, for all $v \in \mathcal{D}(D^{1/2})$ and $w \in \mathcal{D}(D^{1/2})$, we have:

$$\begin{aligned} ((\omega_{ID^{-1/2}v, ID^{1/2}w} * id)(V)w'|v') &= ((id * \omega_{w',v'})(V)D^{1/2}Iv|D^{-1/2}Iw) \\ &= (D^{1/2}I(id * \omega_{v',w'})(V)v|D^{-1/2}Iw) \\ &= (w|(id * \omega_{v',w'})(V)v) \\ &= ((\omega_{v,w} * id)(V)^*w', v') \end{aligned}$$

Then, the proposition holds. □

PROPOSITION 4.49. — *The following relations are satisfied:*

$$\begin{aligned}
i) & (I \underset{N^o}{\alpha} \otimes_{\epsilon} J_{\Phi'}) V = V^* (I \underset{N}{\beta} \otimes_{\alpha} J_{\Phi'}); \\
ii) & (D^{-1} \underset{\nu^o}{\alpha} \otimes_{\epsilon} \Delta_{\Phi'}) V = V (D^{-1} \underset{\nu}{\beta} \otimes_{\alpha} \Delta_{\Phi'}); \\
iii) & (\tau_t \underset{N}{\beta} \star_{\alpha} \sigma_t^{\Phi'}) \circ \Gamma = \Gamma \circ \sigma_t^{\Phi'} \text{ for all } t \in \mathbb{R}.
\end{aligned}$$

where $\epsilon(n) = J_{\Phi'} \alpha(n^*) J_{\Phi'}$ for all $n \in N$.

Proof. — We denote by $S_{\Phi'}$ the operator of Tomita's theory associated with Φ' and defined as the closed operator on $H_{\Phi'}$ such that $\Lambda_{\Phi'}(\mathcal{N}_{\Phi'} \cap \mathcal{N}_{\Phi'}^*)$ is a core for $S_{\Phi'}$ and $S_{\Phi'} \Lambda_{\Phi'}(x) = \Lambda_{\Phi'}(x^*)$ for all $x \in \mathcal{N}_{\Phi'} \cap \mathcal{N}_{\Phi'}^*$. Then, by definition, we have $\Delta_{\Phi'} = S_{\Phi'}^* S_{\Phi'}$ and $S_{\Phi'} = J_{\Phi'} \Delta_{\Phi'}^{1/2}$. Moreover, for all $m \in M$ and $t \in \mathbb{R}$, we have $\sigma_t^{\Phi'}(m) = \Delta_{\Phi'}^{it} m \Delta_{\Phi'}^{-it}$.

First of all, we verify these relations make sense. We have to prove some commutation relations (4.6, 4.21 and [Sau86]). We can write for all $n \in \mathcal{T}_{\nu}$ and $y \in \mathcal{N}_{\Phi'} \cap \mathcal{N}_{\Phi'}^*$:

$$\begin{aligned}
S_{\Phi'} \alpha(n) \Lambda_{\Phi'}(y) &= S_{\Phi'} \Lambda_{\Phi'}(\alpha(n)y) \\
&= \Lambda_{\Phi'}(y^* \alpha(n^*)) = \hat{\alpha}(\sigma_{-i/2}^{\nu}(n)) S_{\Phi'} \Lambda_{\Phi'}(y)
\end{aligned}$$

so $\hat{\alpha}(\sigma_{-i/2}^{\nu}(n)) S_{\Phi'} \subseteq S_{\Phi'} \alpha(n)$ and by adjoint $\alpha(n) S_{\Phi'}^* \subseteq S_{\Phi'}^* \hat{\alpha}(\sigma_{i/2}^{\nu}(n))$. Then:

$$\alpha(n) \Delta_{\Phi'} = \alpha(n) S_{\Phi'}^* S_{\Phi'} \subseteq S_{\Phi'}^* \hat{\alpha}(\sigma_{i/2}^{\nu}(n)) S_{\Phi'} \subseteq \Delta_{\Phi'} \alpha(\sigma_i^{\nu}(n))$$

Since $\beta(n) D^{-1} \subseteq D^{-1} \beta(\sigma_i^{\nu}(n))$, the second relation makes sense. On an other hand, we know that $I \beta(n) = \alpha(n^*) I$ and $\mathcal{J} \alpha(n) = \epsilon(n^*) J_{\Phi'}$ to terms of the first relation. Finally, for all $t \in \mathbb{R}$, we have:

$$\tau_t \circ \beta = \beta \circ \sigma_t^{\nu} \quad \text{and} \quad \sigma_t^{\Phi'}(\alpha(n)) = \Delta_{\Phi'}^{it} \alpha(n) \Delta_{\Phi'}^{-it} = \alpha(\sigma_t^{\nu}(n))$$

which finishes verifications.

Let $v, w \in \Lambda_{\Phi}(\mathcal{T}_{\Phi, S_L})$. By 3.6, we know that $(\omega_{v, w} \underset{\nu}{\beta} \star_{\alpha} id)(\Gamma(y))$ belongs to $\mathcal{N}_{T'} \cap \mathcal{N}_{\Phi'} \cap \mathcal{N}_{T'}^* \cap \mathcal{N}_{\Phi'}^*$ for all $y \in \mathcal{N}_{T'} \cap \mathcal{N}_{\Phi'} \cap \mathcal{N}_{T'}^* \cap \mathcal{N}_{\Phi'}^*$. By 3.18, we can write $(\omega_{v, w} \star id)(V^*) \Lambda_{\Phi'}(y) = \Lambda_{\Phi'}((\omega_{v, w} \underset{\nu}{\beta} \star_{\alpha} id)(\Gamma(y)))$ so that $(\omega_{v, w} \star id)(V^*) \Lambda_{\Phi'}(y)$ belongs to $\mathcal{D}(S_{\Phi'})$. Then, we compute:

$$\begin{aligned}
S_{\Phi'}(\omega_{v, w} \star id)(V^*) \Lambda_{\Phi'}(y) &= S_{\Phi'} \Lambda_{\Phi'}((\omega_{v, w} \underset{\nu}{\beta} \star_{\alpha} id)(\Gamma(y))) \\
&= \Lambda_{\Phi'}((\omega_{v, w} \underset{\nu}{\beta} \star_{\alpha} id)(\Gamma(y^*))) \\
&= (\omega_{w, v} \star id)(V^*) \Lambda_{\Phi'}(y^*) \\
&= (\omega_{w, v} \star id)(V^*) S_{\Phi'} \Lambda_{\Phi'}(y)
\end{aligned}$$

Since $\Lambda_{\Phi'}(\mathcal{N}_{T'} \cap \mathcal{N}_{\Phi'} \cap \mathcal{N}_{T'}^* \cap \mathcal{N}_{\Phi'}^*)$ is a core for $S_{\Phi'}$, this implies:

$$(10) \quad (\omega_{w, v} \star id)(V^*) S_{\Phi'} \subseteq S_{\Phi'}(\omega_{v, w} \star id)(V^*)$$

Take adjoint so as to get:

$$(11) \quad (\omega_{w, v} \star id)(V) S_{\Phi'}^* \subseteq S_{\Phi'}^*(\omega_{v, w} \star id)(V)$$

Then, we deduce by the previous lemma:

$$\begin{aligned} (\omega_{v,w} * id)(V) \Delta_{\Phi'} &= (\omega_{v,w} * id)(V) S_{\Phi'}^* S_{\Phi'} \\ &\subseteq S_{\Phi'}^* (\omega_{v,w} * id)(V) S_{\Phi'} \\ &= S_{\Phi'}^* [(\omega_{ID^{-1/2}w, ID^{1/2}v} * id)(V)]^* S_{\Phi'} \end{aligned}$$

Then by inclusion (10) and the previous lemma, we have:

$$\begin{aligned} (\omega_{v,w} * id)(V) \Delta_{\Phi'} &\subseteq S_{\Phi'}^* S_{\Phi'} [(\omega_{ID^{1/2}v, ID^{-1/2}w} * id)(V)]^* \\ &= \Delta_{\Phi'} (\omega_{D^{1/2}ID^{1/2}v, D^{-1/2}ID^{-1/2}w} * id)(V) \\ &= \Delta_{\Phi'} (\omega_{Dv, N^{-1}w} * id)(V) \end{aligned}$$

Consequently, like relation (8), we easily deduce that:

$$(D^{-1} \underset{\nu^o}{\alpha} \otimes_{\epsilon} \Delta_{\Phi'}) V = V (D^{-1} \underset{\nu}{\beta} \otimes_{\alpha} \Delta_{\Phi'})$$

Let's prove the first relation. By inclusion (10), for all $v \in \mathcal{D}(N^{-1/2})$ and $w \in \mathcal{D}(D^{1/2})$, we have:

$$\begin{aligned} (12) \quad J_{\Phi'}(\omega_{w,v} * id)(V^*) J_{\Phi'} \Delta_{\Phi'}^{1/2} &= J_{\Phi'}(\omega_{w,v} * id)(V^*) S_{\Phi'} \\ &\subseteq J_{\Phi'} S_{\Phi'} (\omega_{v,w} * id)(V^*) \\ &= \Delta_{\Phi'}^{1/2} (\omega_{v,w} * id)(V^*) \end{aligned}$$

For all $p, q \in \mathcal{D}(\Delta_{\Phi'}^{1/2})$, we have by ii):

$$\begin{aligned} ((\omega_{v,w} * id)(V^*) p, \Delta_{\Phi'}^{1/2} q) &= (V^* (v \underset{\nu^o}{\alpha} \otimes_{\epsilon} p) | w \underset{\nu}{\beta} \otimes_{\alpha} \Delta_{\Phi'}^{1/2} q) \\ &= (V^* (v \underset{\nu^o}{\alpha} \otimes_{\epsilon} p) | D^{-1/2} (D^{1/2} w) \underset{\nu}{\beta} \otimes_{\alpha} \Delta_{\Phi'}^{1/2} q) \\ &= ((D^{-1/2} \underset{\nu}{\beta} \otimes_{\alpha} \Delta_{\Phi'}^{1/2}) V^* (v \underset{\nu^o}{\alpha} \otimes_{\epsilon} p) | D^{1/2} w \underset{\nu}{\beta} \otimes_{\alpha} q) \\ &= (V^* (D^{-1/2} v \underset{\nu^o}{\alpha} \otimes_{\epsilon} \Delta_{\Phi'}^{1/2} p) | D^{1/2} w \underset{\nu}{\beta} \otimes_{\alpha} q) \\ &= ((\omega_{D^{-1/2}v, D^{1/2}w} * id)(V^*) \Delta_{\Phi'}^{1/2} p | q). \end{aligned}$$

Since $\Delta_{\Phi'}^{1/2}$ is self-adjoint, we get:

$$(\omega_{D^{-1/2}v, D^{1/2}w} * id)(V^*) \Delta_{\Phi'}^{1/2} \subseteq \Delta_{\Phi'}^{1/2} (\omega_{v,w} * id)(V^*)$$

Also, by the previous lemma, we have:

$$\begin{aligned} (\omega_{D^{-1/2}v, D^{1/2}w} * id)(V^*) &= (\omega_{D^{1/2}w, D^{-1/2}v} * id)(V)^* \\ &= (\omega_{Iw, Iv} * id)(V) \end{aligned}$$

That's why $(\omega_{Iw,Iv} * id)(V) \Delta_{\Phi'}^{1/2} \subseteq \Delta_{\Phi'}^{1/2}(\omega_{v,w} * id)(V^*)$. Since $\Delta_{\Phi'}^{1/2}$ has dense range, this last inclusion and (12) imply that:

$$(\omega_{Iw,Iv} * id)(V) = J_{\Phi'}(\omega_{v,w} * id)(V^*)J_{\Phi'}$$

Then, we can compute:

$$\begin{aligned} & ((I \underset{\nu}{\beta} \otimes_{\alpha} J_{\Phi'}) V^* (I \underset{\nu}{\beta} \otimes_{\alpha} J_{\Phi'})(v \underset{\nu}{\beta} \otimes_{\alpha} q) | w \underset{\nu^o}{\alpha} \otimes_{\epsilon} q) \\ &= (V(Iw \underset{\nu}{\beta} \otimes_{\alpha} J_{\Phi'} q) | Iv \underset{\nu^o}{\alpha} \otimes_{\epsilon} J_{\Phi'} p) \\ &= ((\omega_{Iw,Iv} * id)(V) J_{\Phi'} q | J_{\Phi'} p) = (J_{\Phi'}(\omega_{w,v} * id)(V^*) q | J_{\Phi'} p) \\ &= ((\omega_{v,w} * id)(V) p | q) = (V(v \underset{\nu}{\beta} \otimes_{\alpha} q) | w \underset{\nu^o}{\alpha} \otimes_{\epsilon} q) \end{aligned}$$

so that the first relation is proved. We end the proof by the last equality. We know that Γ is implemented by V , $\sigma^{\Phi'}$ by $\Delta_{\Phi'}$ and τ by D so that the relation comes from $(D^{-1} \underset{\nu}{\alpha} \otimes_{\epsilon} \Delta_{\Phi'}) V = V(D^{-1} \underset{\nu}{\beta} \otimes_{\alpha} \Delta_{\Phi'})$ like 4.27. \square

If we take $T' = T_L$ then $V = W^*$, $J_{\Phi'} = J_{\Phi}$ and $\Delta_{\Phi'} = \Delta_{\Phi}$ so that we have the following propositions:

PROPOSITION 4.50. — *For all $v, w \in \Lambda_{\Phi}(\mathcal{T}_{\Phi, S_L})$, $(id * \omega_{v,w})(W)$ belongs to $\mathcal{D}(S)$ and:*

$$S((id * \omega_{v,w})(W)) = (id * \omega_{v,w})(W^*)$$

PROPOSITION 4.51. — *We have $(\omega_{v,w} * id)(W^*)^* = (\omega_{ID^{-1/2}v, ID^{1/2}w} * id)(W^*)$ for all $v \in \mathcal{D}(D^{1/2})$ and $w \in \mathcal{D}(D^{1/2})$.*

PROPOSITION 4.52. — *The following relations are satisfied:*

- i) $(I \underset{N^o}{\alpha} \otimes_{\beta} J_{\Phi}) W^* = W(I \underset{N}{\beta} \otimes_{\alpha} J_{\Phi})$;
- ii) $(D^{-1} \underset{\nu}{\beta} \otimes_{\alpha} \Delta_{\Phi}) W^* = W^*(D^{-1} \underset{\nu}{\beta} \otimes_{\alpha} \Delta_{\Phi})$;
- iii) $(\tau_t \underset{N}{\beta} \star_{\alpha} \sigma_t^{\Phi}) \circ \Gamma = \Gamma \circ \sigma_t^{\Phi}$ for all $t \in \mathbb{R}$.

We summarize the results of this section in the two following theorems:

THEOREM 4.53. — *Let $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$ be a measured quantum groupoid and W the pseudo-multiplicative unitary associated with. Then the closed linear span of $(id * \omega_{v,w})(W)$ for all $v \in D(\alpha H_{\Phi}, \nu)$ and $w \in D((H_{\Phi})_{\hat{\beta}}, \nu^o)$ is equal to the whole von Neumann algebra M .*

THEOREM 4.54. — *Let $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$ be a measured quantum groupoid and W the pseudo-multiplicative associated with. If we put $\Phi = \nu \circ \alpha^{-1} \circ T_L$, then there exists an unbounded antipode S which satisfies:*

- i) for all $x \in \mathcal{D}(S)$, $S(x)^* \in \mathcal{D}(S)$ and $S(S(x)^*)^* = x$

ii) for all $v, w \in \Lambda_\Phi(\mathcal{T}_{\Phi, S_L})$, $(id * \omega_{v,w})(W)$ belongs to $\mathcal{D}(S)$ and:

$$S((id * \omega_{v,w})(W)) = (id * \omega_{v,w})(W^*)$$

S has the following polar decomposition $S = R\tau_{i/2}$, where R is a co-involution of M satisfying $R^2 = id$, $R \circ \alpha = \beta$ and $\varsigma_{N \circ} \circ (R \underset{N}{\beta \star \alpha} R) \circ \Gamma = \Gamma \circ R$, and where τ , the so-called scaling group, is a one-parameter group of automorphisms such that $\tau_t \circ \alpha = \alpha \circ \sigma_t^\nu$, $\tau_t \circ \beta = \beta \circ \sigma_t^\nu$ satisfying $\Gamma \circ \tau_t = (\tau_t \underset{N}{\beta \star \alpha} \tau_t) \circ \Gamma$ for all $t \in \mathbb{R}$. S, R and τ are independent of T_L and of T_R .

Moreover, $R \circ T_L \circ R$ is a n.s.f operator-valued weight which is right invariant and α -adapted w.r.t ν .

5. Uniqueness, modulus and scaling operator

In this section, the quasi-invariant weight ν is fixed. We establish uniqueness of invariant operator-valued weight which is adapted w.r.t ν up to a strictly positive element affiliated with the center of the basis. We construct a modulus and a scaling operator which link the left invariant operator-valued weight T_L and the right invariant operator-valued weight $R \circ T_L \circ R$. Their properties imply that the fundamental pseudo-multiplicative unitary satisfies a condition similar to Woronowicz's manageability. Also, we study conditions so that a n.s.f weight ν' on N is quasi-invariant.

5.1. Commutation relations. — In this section, we denote by T' an other n.s.f operator-valued weight from M to $\alpha(N)$ which is left invariant and β -adapted w.r.t ν . We put $\Phi' = \nu \circ \alpha^{-1} \circ T'$.

LEMMA 5.1. — *If we put $\kappa_t = \sigma_t^{\Phi'} \circ \tau_{-t}$ for all $t \in \mathbb{R}$, then Ψ is κ -invariant. Also, if we put $\kappa'_t = \sigma_t^\Psi \circ \tau_t$ for all $t \in \mathbb{R}$, then Φ is κ' -invariant.*

Proof. — We know $\kappa_t \circ \alpha = \alpha$ for all $t \in \mathbb{R}$ and we have:

$$\begin{aligned} \Gamma \circ \kappa_t &= \Gamma \circ \sigma_t^{\Phi'} \circ \tau_{-t} = (\tau_t \underset{N}{\beta \star \alpha} \sigma_t^{\Phi'}) \Gamma \circ \tau_{-t} \\ &= (\tau_t \tau_{-t} \underset{N}{\beta \star \alpha} \sigma_t^{\Phi'} \tau_{-t}) \Gamma = (id \underset{N}{\beta \star \alpha} \kappa_t) \Gamma \end{aligned}$$

By right invariance of T_R , we deduce, for all $a \in \mathcal{M}_{T_R}^+$:

$$T_R \circ \kappa_t(a) = (\Psi \underset{\nu}{\beta \star \alpha} id) \Gamma(\kappa_t(a)) = \kappa_t((\Psi \underset{\nu}{\beta \star \alpha} id) \Gamma(a)) = \kappa_t \circ T_R(a)$$

Since T' is β -adapted w.r.t ν , we get for all $a \in \mathcal{M}_{T_R} \cap \mathcal{M}_\Psi^+$ and $t \in \mathbb{R}$:

$$\begin{aligned} \Psi \circ \kappa_t(a) &= \nu \circ \beta^{-1} \circ T_R \circ \kappa_t(a) = \nu \circ \beta^{-1} \circ \kappa_t \circ T_R(a) \\ &= \nu \circ \beta^{-1} \circ \sigma_t^{\Phi'} \circ \tau_{-t} \circ T_R(a) \\ &= \nu \circ \sigma_{-t}^\nu \circ \beta^{-1} \circ \tau_{-t} \circ T_R(a) \\ &= \nu \circ \sigma_{-2t}^\nu \circ T_R(a) = \nu \circ \beta^{-1} \circ T_R(a) = \Psi(a) \end{aligned}$$

The proof of the second part is very similar. \square

PROPOSITION 5.2. — $\sigma^{\Phi'}$ and τ (resp. σ^Ψ and τ) commute each other.

Proof. — Since Ψ is κ -invariant, we know that $\sigma_s^\Psi \circ \sigma_t^{\Phi'} \circ \tau_{-t} = \sigma_t^{\Phi'} \circ \tau_{-t} \circ \sigma_s^\Psi$, for all $s, t \in \mathbb{R}$ so that:

$$\begin{aligned} (id_{\beta \star_\alpha \kappa_t}) \Gamma &= \Gamma \circ \kappa_t = \Gamma \circ \sigma_{-s}^\Psi \circ \kappa_t \circ \sigma_s^\Psi = (\sigma_{-s}^\Psi \beta \star_\alpha \tau_s) \circ \Gamma \circ \kappa_t \circ \sigma_s^\Psi \\ &= (\sigma_{-s}^\Psi \beta \star_\alpha \tau_s \circ \kappa_t) \circ \Gamma \circ \sigma_s^\Psi = (id_{\beta \star_\alpha \tau_s \circ \kappa_t} \circ \sigma_{-s}^\Psi) \circ \Gamma \end{aligned}$$

So, for all $a \in M$, $\omega \in M_*^+$ and $k \in \mathbb{R}^+$ such that $\omega \circ \beta \leq k\nu$, we get:

$$\sigma_t^{\Phi'} \circ \tau_{-t}((\omega \beta \star_\alpha id) \Gamma(a)) = \tau_s \circ \sigma_t^{\Phi'} \circ \tau_{-t} \circ \tau_{-s}((\omega \beta \star_\alpha id) \Gamma(a))$$

and by theorem 4.28, we easily obtain commutation between $\sigma^{\Phi'}$ and τ . A similar reasoning from κ' implies the second part. \square

COROLLARY 5.3. — $\sigma^{\Phi'}$ and σ^Ψ commute each other.

Proof. — By the previous proposition, we compute, for all $s, t \in \mathbb{R}$:

$$\begin{aligned} \Gamma \circ \sigma_s^{\Phi'} \circ \sigma_t^\Psi &= (\tau_s \beta \star_\alpha \sigma_s^{\Phi'}) \circ \Gamma \circ \sigma_t^\Psi = (\tau_s \sigma_t^\Psi \beta \star_\alpha \sigma_s^{\Phi'} \tau_{-t}) \circ \Gamma \\ &= (\sigma_t^\Psi \tau_s \beta \star_\alpha \tau_{-t} \sigma_s^{\Phi'}) \circ \Gamma \\ &= (\sigma_t^\Psi \beta \star_\alpha \tau_{-t}) \circ \Gamma \circ \sigma_s^{\Phi'} = \Gamma \circ \sigma_t^\Psi \circ \sigma_s^{\Phi'} \end{aligned}$$

Since Γ is injective, we have done. \square

5.2. First result about uniqueness of invariant operator-valued weight. — In this section, we choose to work with left invariant operator-valued weights, but it is clear that we have similar results for right invariant operator-valued weights. Let T_1 and T_2 be two n.s.f left invariant operator-valued weights from M to $\alpha(N)$ such that $T_1 \leq T_2$. For all $i \in \{1, 2\}$, we put $\Phi_i = \nu \circ \alpha^{-1} \circ T_i$ and $\hat{\beta}_i(n) = J_{\Phi_i} \alpha(n^*) J_{\Phi_i}$.

We define, as we have done for U_H , an isometry $(U_2)_H$ by the following formula:

$$(U_2)_H(v \underset{\nu^o}{\alpha \otimes \hat{\beta}_2} \Lambda_{\Phi_2}(a)) = \sum_{i \in I} \xi_i \underset{\nu}{\beta \otimes \alpha} \Lambda_{\Phi_2}((\omega_{v, \xi_i} \underset{\nu}{\beta \star \alpha} id)(\Gamma(a)))$$

for all $v \in D(H_\beta, \nu^o)$ and $a \in \mathcal{N}_{\Phi_2} \cap \mathcal{N}_{T_2}$. Then, we know that $(U_2)_H$ is unitary and $\Gamma(m) = (U_2)_H(1 \underset{N^o}{\alpha \otimes \hat{\beta}_2} m)(U_2)_H^*$ for all $m \in M$.

Since $T_1 \leq T_2$, there exists $F \in \mathcal{L}(H_{\Phi_2}, H_{\Phi_1})$ such that, for all $x \in \mathcal{N}_{\Phi_2} \cap \mathcal{N}_{T_2}$, we have $F\Lambda_{\Phi_2}(x) = \Lambda_{\Phi_1}(x)$. It is easy to verify that, for all $n \in N$, we have $F\hat{\beta}_2(n) = \hat{\beta}_1(n)F$. If we put $P = F^*F$, then P belongs to $M' \cap \hat{\beta}_2(N)'$ and $J_{\Phi_2}PJ_{\Phi_2}$ belongs to $M \cap \alpha(N)'$.

LEMMA 5.4. — We have $\Gamma(J_{\Phi_2}PJ_{\Phi_2}) = 1 \underset{N}{\beta \otimes \alpha} J_{\Phi_2}PJ_{\Phi_2}$.

Proof. — We have, for all $v, w \in D(H_\beta, \nu^o)$ and $a, b \in \mathcal{N}_{\Phi_2} \cap \mathcal{N}_{T_2}$:

$$\begin{aligned} & ((1 \underset{N}{\beta \otimes \alpha} P)(U_2)_H(v \underset{\nu^o}{\alpha \otimes \hat{\beta}_2} \Lambda_{\Phi_2}(a)) | (U_2)_H(w \underset{\nu^o}{\alpha \otimes \hat{\beta}_2} \Lambda_{\Phi_2}(b))) \\ &= ((U_1)_H(v \underset{\nu^o}{\alpha \otimes \hat{\beta}_1} \Lambda_{\Phi_1}(a)) | (U_1)_H(w \underset{\nu^o}{\alpha \otimes \hat{\beta}_1} \Lambda_{\Phi_1}(b))) \end{aligned}$$

where $(U_1)_H$ is defined in the same way as $(U_2)_H$. The two expressions are continuous in v and w , so by density of $D(H_\beta, \nu^o)$ in H , we get, for all $v, w \in H$ and $a, b \in \mathcal{N}_{\Phi_2} \cap \mathcal{N}_{T_2}$:

$$\begin{aligned} & ((1 \underset{N}{\beta \otimes \alpha} P)(U_2)_H(v \underset{\nu^o}{\alpha \otimes \hat{\beta}_2} \Lambda_{\Phi_2}(a)) | (U_2)_H(w \underset{\nu^o}{\alpha \otimes \hat{\beta}_2} \Lambda_{\Phi_2}(b))) \\ &= ((U_1)_H(v \underset{\nu^o}{\alpha \otimes \hat{\beta}_1} \Lambda_{\Phi_1}(a)) | (U_1)_H(w \underset{\nu^o}{\alpha \otimes \hat{\beta}_1} \Lambda_{\Phi_1}(b))) \\ &= (v \underset{\nu^o}{\alpha \otimes \hat{\beta}_1} \Lambda_{\Phi_1}(a) | w \underset{\nu^o}{\alpha \otimes \hat{\beta}_1} \Lambda_{\Phi_1}(b)) \\ &= ((1 \underset{N^o}{\alpha \otimes \hat{\beta}_2} P)(v \underset{\nu^o}{\alpha \otimes \hat{\beta}_2} \Lambda_{\Phi_2}(a)) | w \underset{\nu^o}{\alpha \otimes \hat{\beta}_2} \Lambda_{\Phi_2}(b)) \end{aligned}$$

so that $(U_2)_H^*(1 \underset{N}{\beta \otimes \alpha} P)(U_2)_H = 1 \underset{N^o}{\alpha \otimes \hat{\beta}_2} P$. In particular, if $H = H_\Phi$, then by 4.49 we get $(U_2)_H(1 \underset{N^o}{\alpha \otimes \hat{\beta}_2} J_{\Phi_2}PJ_{\Phi_2})(U_2)_H^* = 1 \underset{N}{\beta \otimes \alpha} J_{\Phi_2}PJ_{\Phi_2}$. Finally, since $J_{\Phi_2}PJ_{\Phi_2} \in M$, we have $\Gamma(J_{\Phi_2}PJ_{\Phi_2}) = 1 \underset{N}{\beta \otimes \alpha} J_{\Phi_2}PJ_{\Phi_2}$. \square

PROPOSITION 5.5. — If T_1 and T_2 are n.s.f left invariant weights from M to $\alpha(N)$ such that $T_1 \leq T_2$, then there exists an injective $p \in N$ such that $0 \leq p \leq 1$ and, for all $x, y \in \mathcal{N}_{\Phi_2} \cap \mathcal{N}_{T_2}$, we have $(\Lambda_{\Phi_1}(x) | \Lambda_{\Phi_1}(y)) = (J_{\Phi_2}\beta(p)J_{\Phi_2}\Lambda_{\Phi_2}(x) | \Lambda_{\Phi_2}(y))$.

Proof. — Straightforward from the previous lemma and 4.33. \square

PROPOSITION 5.6. — *Let T_1 and T_2 be n.s.f left invariant weights from M to $\alpha(N)$ such that $T_1 \leq T_2$. If T_1 and T_2 are β -adapted w.r.t ν , then there exists an injective $p \in Z(N)$ such that $0 \leq p \leq 1$ and $\Phi_1 = (\Phi_2)_{\beta(p)}$ in the sense of [Str81]. (We recall that $\Phi_i = \nu \circ \alpha^{-1} \circ T_i$ for $i = 1, 2$).*

Proof. — Since T_1 and T_2 are β -adapted w.r.t ν , σ^{T_1} and σ^{T_2} are equal on $\beta(N)$. For all $x, y \in \mathcal{N}_{\Phi_2} \cap \mathcal{N}_{T_2}$ and $n \in \mathcal{T}_\nu$, we compute:

$$\begin{aligned} (PJ_{\Phi_2}\beta(n)J_{\Phi_2}\Lambda_{\Phi_2}(x)|\Lambda_{\Phi_2}(y)) &= (P\Lambda_{\Phi_2}(x\sigma_{-i/2}^{T_2}(\beta(n))|\Lambda_{\Phi_2}(y)) \\ &= (\Lambda_{\Phi_1}(x\sigma_{-i/2}^{T_1}(\beta(n))|\Lambda_{\Phi_1}(y)) \\ &= \Phi_1(y^*x\sigma_{-i/2}^{T_1}(\beta(n))) \end{aligned}$$

Then K.M.S conditions, applied for the n.s.f Φ_1 on M , imply that the last expression is equal to:

$$\begin{aligned} \Phi_1(\sigma_{i/2}^{T_1}(\beta(n))y^*x) &= (\Lambda_{\Phi_1}(x)|\Lambda_{\Phi_1}(y\sigma_{-i/2}^{T_1}(\beta(n^*)))) \\ &= (P\Lambda_{\Phi_2}(x)|\Lambda_{\Phi_2}(y\sigma_{-i/2}^{T_2}(\beta(n^*)))) \\ &= (J_{\Phi_2}\beta(n)J_{\Phi_2}P\Lambda_{\Phi_2}(x)|\Lambda_{\Phi_2}(y)) \end{aligned}$$

That's why $P \in (J_{\Phi_2}\beta(N)J_{\Phi_2})'$ and, consequently, by the previous proposition and injectivity of β , we get $p \in Z(N)$. We know that $0 \leq P \leq 1$ and P is injectif with dense range, so the same is for p . Finally, by the previous proposition and [Str81] (proposition 3.13), we get that $\beta(p)$ coincides with the analytic continuation in $-i$ of the cocycle $[D\Phi_1 : D\Phi_2]$. Then, we have:

$$[D\Phi_1 : D\Phi_2]_t = \beta(p)^{it}$$

for all $t \in \mathbb{R}$. Since $p \in Z(N)$ and T_2 is β -adapted w.r.t ν , we have $\beta(p)$ belongs to the centralizer of Φ_2 and $\Phi_1 = (\Phi_2)_{\beta(p)}$. \square

LEMMA 5.7. — *Let T and T' be n.s.f operator-valued weights which are β -adapted w.r.t ν . If $T + T'$ is semi-finite, then $T + T'$ is β -adapted w.r.t ν .*

Proof. — Since T and T' are β -adapted w.r.t ν , by 4.4, there exists n.s.f operator-valued weights S from M to $\beta(N)$ such that:

$$\nu \circ \alpha^{-1} \circ T = \nu \circ \beta^{-1} \circ S \quad \text{and} \quad \nu \circ \alpha^{-1} \circ T' = \nu \circ \beta^{-1} \circ S'$$

Consequently $\nu \circ \alpha^{-1} \circ (T + T') = \nu \circ \beta^{-1} \circ (S + S')$. This weight is semi-finite, since $T + T'$ is. Then $S + S'$ is also semi-finite. We deduce, for all $n \in N$ and $t \in \mathbb{R}$:

$$\begin{aligned} \sigma_t^{T+T'}(\beta(n)) &= \sigma_t^{\nu \circ \alpha^{-1} \circ (T+T')}(\beta(n)) = \sigma_t^{\nu \circ \beta^{-1} \circ (S+S')}(\beta(n)) \\ &= \sigma_t^{\nu \circ \beta^{-1}}(\beta(n)) = \beta(\sigma_{-t}^\nu(n)) \end{aligned}$$

\square

We recall a technical lemma of [Kus97]:

LEMMA 5.8. — *If ϕ and η are n.s.f weights on M and if there exists a strictly positive operator λ which is affiliated with $(M^\phi)^+$ satisfying:*

$$\|\Lambda_\eta(\sigma_t^\phi(x))\| = \|\lambda^{\frac{t}{2}}\Lambda_\eta(x)\|$$

for all $x \in \mathcal{N}_\eta$ and $t \in \mathbb{R}$, then $\mathcal{N}_\eta \cap \mathcal{N}_\phi$ is a core for both Λ_η and Λ_ϕ .

Proof. — We can define unitary T_t such that $T_t\Lambda_\eta(a) = \lambda^{-t/2}\Lambda_\eta(\sigma_t^\phi(a))$ for all $t \in \mathbb{R}$. Moreover, there exists a strictly positive operator T such that $T^{it} = T_t$ for all $t \in \mathbb{R}$. The end of the proof is similar to [KV99] (proposition 1.14). \square

PROPOSITION 5.9. — *Let T_1 be a n.s.f left invariant operator-valued weight, which is β -adapted w.r.t ν , such that there exists a strictly positive operator λ which is affiliated to $(M^\Phi)^+$ satisfying $\|\Lambda_1(\sigma_t^\Phi(x))\| = \|\lambda^{\frac{t}{2}}\Lambda_1(x)\|$ for all $x \in \mathcal{N}_{\Phi_1}$ and $t \in \mathbb{R}$. Then, there exists a strictly positive operator q which is affiliated to the center of N such that $\Phi_1 = (\Phi)_{\beta(q)}$.*

Proof. — We put $T_2 = T_L + T_1$. Since $\|\Lambda_1(\sigma_t^\Phi(x))\| = \|\lambda^{\frac{t}{2}}\Lambda_1(x)\|$ for all $x \in \mathcal{N}_{\Phi_1}$ and $t \in \mathbb{R}$, the left invariant operator-valued weight T_2 is n.s.f. So, by 5.7, T_2 is β -adapted w.r.t ν . Finally, since $T_1 \leq T_2$ and $T_L \leq T_2$, by 5.6, there exists an injective $p \in N$ between 0 and 1 such that $\Phi_1 = (\Phi_2)_{\beta(p)}$ and $\Phi = (\Phi_2)_{\beta(1-p)}$. By [Str81], we have:

$$[D\Phi_1 : D\Phi_2]_t = \beta(p)^{it} \text{ and } [D\Phi : D\Phi_2]_t = \beta(1-p)^{it}$$

Then, we have, for all $t \in \mathbb{R}$:

$$[D\Phi_1 : D\Phi]_t = [D\Phi_1 : D\Phi_2]_t [D\Phi_2 : D\Phi]_t = \beta\left(\frac{p}{1-p}\right)^{it}$$

that's why $q = \frac{p}{1-p}$ is the suitable element. \square

5.3. Modulus and scaling operator. — From now, we study the following measured quantum group $(N, M, \alpha, \beta, \Gamma, \nu, T_L, R \circ T_L \circ R)$ so that we look at $\Phi = \nu \circ \alpha^{-1} \circ T_L$ and $\Phi \circ R$. Then, we recall $\sigma_t^{\Phi \circ R} = R \circ \sigma_{-t}^\Phi \circ R$ for all $t \in \mathbb{R}$.

By 5.3, we know that modular groups associated with Φ and $\Phi \circ R$ commute each other and, by [Vae01a] (proposition 2.5), there exist a strictly positive operator δ affiliated with M and a strictly positive operator λ affiliated to the center of M such that, for all $t \in \mathbb{R}$, we have $[D\Phi \circ R : D\Phi]_t = \lambda^{\frac{1}{2}it^2}\delta^{it}$. Modular groups of Φ and $\Phi \circ R$ are linked by $\sigma_t^{\Phi \circ R}(m) = \delta^{it}\sigma_t^\Phi(m)\delta^{-it}$ for all $t \in \mathbb{R}$ and $m \in M$.

DEFINITION 5.10. — We call **scaling operator** the strictly positive operator λ affiliated to $Z(M)$ and **modulus** the strictly positive operator δ affiliated to M such that, for all $t \in \mathbb{R}$, we have:

$$[D\Phi \circ R : D\Phi]_t = \lambda^{\frac{1}{2}it^2}\delta^{it}$$

In this section, we establish properties of scaling operator and modulus e.g compatibility of these objects with the Hopf bimodule structure.

PROPOSITION 5.11. — *The scaling operator does not depend on the quasi-invariant weight but just on the modular group associated with. If δ is the class of δ for the equivalent relation $\delta_1 \sim \delta_2$ if, and only if there exists a strictly positive operator h affiliated to $Z(N)$ such that $\delta_2^{it} = \beta(h^{it})\delta_1^{it}\alpha(h^{-it})$, then δ does not depend on the quasi-invariant weight but just on the modular group associated with.*

Proof. — If ν' is a n.s.f weight on N such that $\sigma^{\nu'} = \sigma^\nu$, then there exists a strictly positive h affiliated to $Z(N)$ such that $\nu' = \nu_h$. We just have to compute:

$$\begin{aligned} & [D\nu' \circ \alpha^{-1} \circ T_L \circ R : D\nu' \circ \alpha^{-1} \circ T_L]_t \\ &= [D\nu_h \circ \alpha^{-1} \circ T_L \circ R : D\Phi \circ R]_t [D\Phi \circ R : D\Phi]_t [D\Phi : D\nu_h \circ \alpha^{-1} \circ T_L]_t \\ &= \beta([D\nu_h : D\nu]_{-t}^*) \lambda^{\frac{1}{2}it^2} \delta^{it} \alpha([D\nu : D\nu_h]_t) = \lambda^{\frac{1}{2}it^2} \beta(h^{it}) \delta^{it} \alpha(h^{-it}) \end{aligned}$$

□

LEMMA 5.12. — *For all $s, t \in \mathbb{R}$, we have $[D\Phi \circ \sigma_s^{\Phi \circ R} : D\Phi]_t = \lambda^{ist}$.*

Proof. — The computation of the cocycle is straightforward:

$$\begin{aligned} [D\Phi \circ \sigma_s^{\Phi \circ R} : D\Phi]_t &= [D\Phi \circ \sigma_s^{\Phi \circ R} : D\Phi \circ R \circ \sigma_s^{\Phi \circ R}]_t [D\Phi \circ R : D\Phi]_t \\ &= \sigma_{-s}^{\Phi \circ R}([D\Phi : D\Phi \circ R]_t) [D\Phi \circ R : D\Phi]_t \\ &= \delta^{-is} \sigma_{-s}^{\Phi} (\lambda^{-\frac{it^2}{2}} \delta^{-it}) \delta^{is} \lambda^{\frac{it^2}{2}} \delta^{it} \\ &= \delta^{-is} \lambda^{-\frac{it^2}{2}} \lambda^{ist} \delta^{-it} \delta^{is} \lambda^{\frac{it^2}{2}} \delta^{it} = \lambda^{ist} \end{aligned}$$

□

PROPOSITION 5.13. — *We have $R(\lambda) = \lambda$, $R(\delta) = \delta^{-1}$ and $\tau_t(\delta) = \delta$, $\tau_t(\lambda) = \lambda$ for all $t \in \mathbb{R}$.*

Proof. — Relations between R , λ and δ come from uniqueness of Radon-Nikodym cocycle decomposition. By 5.1, we have $\Phi \circ \tau_{-s} = \Phi \circ \sigma_s^{\Phi \circ R}$ for all $s, t \in \mathbb{R}$, so:

$$\tau_s([D\Phi \circ R : D\Phi]_t) = [D\Phi \circ R \circ \tau_{-s} : D\Phi \circ \tau_{-s}]_t = [D\Phi \circ \sigma_s^{\Phi \circ R} \circ R : D\Phi \circ \sigma_s^{\Phi \circ R}]_t$$

Consequently, by the previous lemma, we get:

$$\begin{aligned} & \tau_s([D\Phi \circ R : D\Phi]_t) \\ &= [D\Phi \circ \sigma_s^{\Phi \circ R} \circ R : D\Phi \circ R]_t [D\Phi \circ R : D\Phi]_t [D\Phi : D\Phi \circ \sigma_s^{\Phi \circ R}]_t \\ &= R([D\Phi \circ \sigma_s^{\Phi \circ R} : D\Phi]_{-t}^*) [D\Phi \circ R : D\Phi]_t [D\Phi \circ \sigma_s^{\Phi \circ R} : D\Phi]_t^* \\ &= R(\lambda^{ist}) \lambda^{-\frac{it^2}{2}} \delta^{it} \lambda^{-ist} = \lambda^{-\frac{it^2}{2}} \delta^{it} \end{aligned}$$

□

COROLLARY 5.14. — *The modulus δ is affiliated with $M \cap \alpha(N)' \cap \beta(N)'$.*

Proof. — Since $\Phi = \nu \circ \beta^{-1} \circ S_L$, we have:

$$\lambda^{\frac{it^2}{2}} \delta^{it} = [D\Phi \circ R : D\Phi]_t = [DR \circ T_L \circ R : DS_L]_t$$

which belongs to $M \cap \beta(N)'$. Since λ is affiliated with $Z(M)$, we get that δ is affiliated with $M \cap \beta(N)'$. Finally, since $R(\delta) = \delta$, we obtain that δ is affiliated with $M \cap \alpha(N)' \cap \beta(N)'$. □

LEMMA 5.15. — *For all $t \in \mathbb{R}$, $\tau_{-t} \circ T_L \circ \tau_t$ is a n.s.f left invariant operator-valued weight from M to $\alpha(N)$. Moreover, $\tau_{-t} \circ T_L \circ \tau_t$ is β -adapted for ν .*

Proof. — For all $t \in \mathbb{R}$, we have $\nu \circ \alpha^{-1} \circ \tau_{-t} \circ T_L \circ \tau_t = \Phi \circ \tau_t$. Then:

$$\begin{aligned} (id_{\beta \star_{\alpha} \nu} \nu \circ \alpha^{-1} \circ \tau_{-t} \circ T_L \circ \tau_t) \circ \Gamma &= (id_{\beta \star_{\alpha} \nu} \Phi \circ \tau_t) \circ \Gamma \\ &= \tau_{-t} \circ (id_{\beta \star_{\alpha} \nu} \Phi) \circ \Gamma \circ \tau_t = \tau_{-t} \circ T_L \circ \tau_t \end{aligned}$$

On the other hand, for all $s, t \in \mathbb{R}$ and $n \in N$, we have:

$$\begin{aligned} \sigma_s^{\Phi \circ \tau_t}(\beta(n)) &= \tau_{-t} \circ \sigma_s^{\Phi} \circ \tau_t(\beta(n)) = \tau_{-t} \circ \sigma_s^{\Phi}(\beta(\sigma_{-t}^{\nu}(n))) \\ &= \tau_{-t}(\beta(\sigma_{-(s+t)}^{\nu}(n))) = \beta(\sigma_{-s}^{\nu}(n)) \end{aligned}$$

□

PROPOSITION 5.16. — *There exists a strictly positive operator q affiliated with $Z(N)$ such that the scaling operator $\lambda = \alpha(q) = \beta(q)$. In particular, λ is affiliated with $Z(M) \cap \alpha(N) \cap \beta(N)$.*

Proof. — By the previous lemma, $\tau_s \circ T_L \circ \tau_{-s}$ is left invariant and β -adapted w.r.t ν . Moreover, since σ^{Φ} and τ commute, $\Phi \circ \tau_{-s}$ is σ^{Φ} -invariant. That's why, we are in 5.9 conditions so that we get a strictly positive operator q_s affiliated with $Z(N)$ such that $[D\Phi \circ \tau_{-s} : D\Phi]_t = \beta(q_s)^{it}$. On the other hand, by 5.12, we have $[D\Phi \circ \sigma_s^{\Phi \circ R} : D\Phi]_t = \lambda^{ist}$. By 5.1, we have $\Phi \circ \tau_{-s} = \Phi \circ \sigma_s^{\Phi \circ R}$, so we obtain that $\lambda^{ist} = \beta(q_s)^{it}$ for all $s, t \in \mathbb{R}$. We easily deduce that there exists a strictly positive operator q affiliated with $Z(N)$ such that $\lambda = \beta(q)$. Finally, since $R(\lambda) = \lambda$, we also have $\lambda = \alpha(q)$. □

LEMMA 5.17. — *We have, for all $a, b \in \mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L}$ and $t \in \mathbb{R}$:*

$$\omega_{J_{\Phi} \Lambda_{\Phi}(\lambda^{\frac{t}{2}} \tau_t(a))} = \omega_{J_{\Phi} \Lambda_{\Phi}(a)} \circ \tau_{-t} \text{ and } \omega_{J_{\Phi} \Lambda_{\Phi}(b)} \circ \sigma_t^{\Phi \circ R} = \omega_{J_{\Phi} \Lambda_{\Phi}(\lambda^{\frac{t}{2}} \sigma_{-t}^{\Phi \circ R}(b))}$$

Proof. — τ is implemented by P that's why the first relation holds. By [Vae01a] (proposition 2.4), we know that $\Delta_{\Phi \circ R} = J_{\Phi} \delta J_{\Phi} \delta \Delta_{\Phi}$ so that we can compute:

$$\begin{aligned}
& (\sigma_t^{\Phi \circ R}(x) J_{\Phi} \Lambda_{\Phi}(b) | J_{\Phi} \Lambda_{\Phi}(b)) \\
&= (x \Delta_{\Phi \circ R}^{-it} J_{\Phi} \Lambda_{\Phi}(b) | \Delta_{\Phi \circ R}^{-it} J_{\Phi} \Lambda_{\Phi}(b)) \\
&= (x J_{\Phi} \delta^{it} J_{\Phi} \delta^{-it} \Delta_{\Phi}^{-it} J_{\Phi} \Lambda_{\Phi}(b) | J_{\Phi} \delta^{it} J_{\Phi} \delta^{-it} \Delta_{\Phi}^{-it} J_{\Phi} \Lambda_{\Phi}(b)) \\
&= (x \delta^{-it} J_{\Phi} \Lambda_{\Phi}(\sigma_{-t}^{\Phi}(b)) | \delta^{-it} J_{\Phi} \Lambda_{\Phi}(\sigma_{-t}^{\Phi}(b))) \\
&= (x J_{\Phi} \Lambda_{\Phi}(\lambda^{\frac{t}{2}} \sigma_{-t}^{\Phi}(b) \delta^{it}) | J_{\Phi} \Lambda_{\Phi}(\lambda^{\frac{t}{2}} \sigma_{-t}^{\Phi}(b) \delta^{it})) \\
&= (x J_{\Phi} \Lambda_{\Phi}(\lambda^{\frac{t}{2}} \delta^{-it} \sigma_{-t}^{\Phi}(b) \delta^{it}) | J_{\Phi} \Lambda_{\Phi}(\lambda^{\frac{t}{2}} \delta^{-it} \sigma_{-t}^{\Phi}(b) \delta^{it})) \\
&= (x J_{\Phi} \Lambda_{\Phi}(\lambda^{\frac{t}{2}} \sigma_{-t}^{\Phi \circ R}(b)) | J_{\Phi} \Lambda_{\Phi}(\lambda^{\frac{t}{2}} \sigma_{-t}^{\Phi \circ R}(b)))
\end{aligned}$$

□

PROPOSITION 5.18. — We have $\Gamma \circ \tau_t = (\sigma_t^{\Phi} \underset{N}{\beta} \star_{\alpha} \sigma_{-t}^{\Phi \circ R}) \circ \Gamma$ for all $t \in \mathbb{R}$.

Proof. — For all $a, b \in \mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L}$ and $t \in \mathbb{R}$, we compute:

$$\begin{aligned}
& (id \underset{\nu}{\beta} \star_{\alpha} \omega_{J_{\Phi} \Lambda_{\Phi}(b)}) [(\sigma_{-t}^{\Phi} \underset{N}{\beta} \star_{\alpha} \sigma_t^{\Phi \circ R}) \circ \Gamma \circ \tau_t(a^* a)] \\
&= \sigma_{-t}^{\Phi} [(id \underset{\nu}{\beta} \star_{\alpha} \omega_{J_{\Phi} \Lambda_{\Phi}(b)} \circ \sigma_t^{\Phi \circ R})(\Gamma \circ \tau_t(a^* a))]
\end{aligned}$$

By the previous lemma, this last expression is equal to:

$$\begin{aligned}
& \sigma_{-t}^{\Phi} [(id \underset{\nu}{\beta} \star_{\alpha} \omega_{J_{\Phi} \Lambda_{\Phi}(\lambda^{\frac{t}{2}} \sigma_{-t}^{\Phi \circ R}(b))})(\Gamma \circ \tau_t(a^* a))] \\
&= \sigma_{-t}^{\Phi} \circ R [(id \underset{\nu}{\beta} \star_{\alpha} \omega_{J_{\Phi} \Lambda_{\Phi}(\tau_t(a))})(\Gamma(\lambda^{\frac{t}{2}} \sigma_{-t}^{\Phi \circ R}(b^* b)))] \\
&= R \circ \sigma_t^{\Phi \circ R} [(id \underset{\nu}{\beta} \star_{\alpha} \omega_{J_{\Phi} \Lambda_{\Phi}(\lambda^{\frac{t}{2}} \tau_t(a))})(\Gamma \circ \sigma_{-t}^{\Phi \circ R}(b^* b))]
\end{aligned}$$

Again, by the previous lemma, this last expression is equal to:

$$\begin{aligned}
& R \circ \sigma_t^{\Phi \circ R} [(id \underset{\nu}{\beta} \star_{\alpha} \omega_{J_{\Phi} \Lambda_{\Phi}(a)} \circ \tau_{-t}(\Gamma \circ \sigma_{-t}^{\Phi \circ R}(b^* b)))] \\
&= R [(id \underset{\nu}{\beta} \star_{\alpha} \omega_{J_{\Phi} \Lambda_{\Phi}(a)})(\Gamma(b^* b))] = (id \underset{\nu}{\beta} \star_{\alpha} \omega_{J_{\Phi} \Lambda_{\Phi}(b)})(\Gamma(a^* a))
\end{aligned}$$

So, we conclude that $(\sigma_{-t}^{\Phi} \underset{N}{\beta} \star_{\alpha} \sigma_t^{\Phi \circ R}) \circ \Gamma \circ \tau_t = \Gamma$ for all $t \in \mathbb{R}$.

□

Since δ is affiliated with $M \cap \alpha(N)' \cap \beta(N)'$, we can define an operator $\delta^{it} \underset{N}{\beta} \otimes_{\alpha} \delta^{it}$ which belongs to $(M \cap \beta(N)') \underset{N}{\beta} \otimes_{\alpha} (M \cap \alpha(N)') \subset M \underset{N}{\beta} \star_{\alpha} M$ for all $t \in \mathbb{R}$. Now, we prove that δ is a group-like element i.e $\Gamma(\delta) = \delta \underset{N}{\beta} \otimes_{\alpha} \delta$.

LEMMA 5.19. — For all $s, t \in \mathbb{R}$, $\Gamma(\delta^{is})$ and $\delta^{it} \underset{N}{\beta} \otimes_{\alpha} \delta^{it}$ commute each other.

Proof. — For all $t \in \mathbb{R}$, we have:

$$\begin{aligned}
 (\sigma_{-t}^\Phi \circ \sigma_t^{\Phi \circ R} \underset{N}{\beta \star_\alpha} \sigma_{-t}^\Phi \circ \sigma_t^{\Phi \circ R}) \circ \Gamma &= (\sigma_{-t}^\Phi \underset{N}{\beta \star_\alpha} \sigma_{-t}^\Phi \circ \sigma_t^{\Phi \circ R} \circ \tau_t) \circ \Gamma \circ \sigma_t^{\Phi \circ R} \\
 &= (\sigma_{-t}^\Phi \circ \tau_t \underset{N}{\beta \star_\alpha} \sigma_t^{\Phi \circ R} \circ \tau_t) \circ \Gamma \circ \sigma_{-t}^\Phi \circ \sigma_t^{\Phi \circ R} \\
 &= (\sigma_{-t}^\Phi \underset{N}{\beta \star_\alpha} \sigma_t^{\Phi \circ R}) \circ \Gamma \circ \tau_t \sigma_{-t}^\Phi \circ \sigma_t^{\Phi \circ R} \\
 &= \Gamma \circ \sigma_{-t}^\Phi \circ \sigma_t^{\Phi \circ R}
 \end{aligned}$$

We know that $\sigma_{-t}^\Phi \sigma_t^{\Phi \circ R}(m) = \delta^{it} m \delta^{-it}$ for all $m \in M$, that's why we get:

$$(\delta^{it} \underset{N}{\beta \otimes_\alpha} \delta^{it}) \Gamma(m) (\delta^{-it} \underset{N}{\beta \otimes_\alpha} \delta^{-it}) = \Gamma(\delta^{it} m \delta^{-it})$$

In particular, for all $s \in \mathbb{R}$, we have:

$$(\delta^{it} \underset{N}{\beta \otimes_\alpha} \delta^{it}) \Gamma(\delta^{is}) (\delta^{-it} \underset{N}{\beta \otimes_\alpha} \delta^{-it}) = \Gamma(\delta^{it} \delta^{is} \delta^{-it}) = \Gamma(\delta^{is})$$

□

We recall [KV00] (result 7.6) in the following lemma:

LEMMA 5.20. — *There exists a subspace C of M such that, for all $c \in C$:*

- i) $c \in \mathcal{T}_{\Phi \circ R}$;
- ii) $\delta^{-1/2} \sigma_{-i/2}^{\Phi \circ R}(c^*)$ is bounded and belongs to $\mathcal{D}(\sigma_{i/2}^{\Phi \circ R}) \cap \mathcal{N}_{\Phi \circ R}$;
- iii) $\delta^{-1/2} c^*$ is bounded and belongs to $\mathcal{N}_{\Phi \circ R}^*$;
- iv) $\sigma_{i/2}^{\Phi \circ R}(\delta^{-1/2} \sigma_{-i/2}^{\Phi \circ R}(c^*)) = \lambda^{-i/4} \delta^{-1/2} c^*$;
- v) $\Lambda_{\Phi \circ R}(C)$ is a core for $\delta^{-1/2}$;
- vi) $c \in \mathcal{N}_\Phi$, $\Phi(c^*c) = \Phi \circ R((\delta^{-1/2} \sigma_{-i/2}^{\Phi \circ R}(c^*))^* \delta^{-1/2} \sigma_{-i/2}^{\Phi \circ R}(c^*))$;
- vii) $T_L(c^*c) = S_R(\delta^{-1/2} c^* c \delta^{-1/2})$

(We recall that S_R satisfies $\nu \circ \beta^{-1} \circ T_R = \Phi \circ R = \nu \circ \alpha^{-1} \circ S_R$).

PROPOSITION 5.21. — *If S_R is the unique n.s.f operator-valued weight which satisfies $\nu \circ \beta^{-1} \circ T_R = \Phi \circ R = \nu \circ \alpha^{-1} \circ S_R$, then we have:*

$$\Phi \circ R((\omega_{\underset{\nu}{v} \beta \star_\alpha} id)(\Gamma(a))) = (S_R(a) \delta^{-1/2} v | \delta^{-1/2} v)$$

for all $v \in \mathcal{D}(\delta^{-1/2}) \cap D((H_{\Phi \circ R})_\beta, \nu^o)$ and $a \in \mathcal{N}_{\Phi \circ R} \cap \mathcal{N}_{S_R}$.

Proof. — Let $c \in C$ and $d \in \mathcal{N}_{\Phi \circ R} \cap \mathcal{N}_{T_R}$. By left invariance of T_L , we have:

$$\begin{aligned}
 \Phi \circ R((\omega_{\underset{\nu}{J_{\Phi \circ R} \Lambda_{\Phi \circ R}(c)} \beta \star_\alpha} id)(\Gamma(d^*d))) &= \Phi((\omega_{\underset{\nu}{J_{\Phi \circ R} \Lambda_{\Phi \circ R}(d)} \beta \star_\alpha} id)(\Gamma(c^*c))) \\
 &= (T_L(c^*c) J_{\Phi \circ R} \Lambda_{\Phi \circ R}(d) | J_{\Phi \circ R} \Lambda_{\Phi \circ R}(d))
 \end{aligned}$$

By properties of elements of C , this last expression is equal to:

$$(S_R(\delta^{-1/2} c^* c \delta^{-1/2}) J_{\Phi \circ R} \Lambda_{\Phi \circ R}(d) | J_{\Phi \circ R} \Lambda_{\Phi \circ R}(d))$$

If we denote by ε the anti-representation of N such that $\varepsilon(n) = J_{\Phi \circ R} \alpha(n^*) J_{\Phi \circ R}$ for all $n \in N$, then the expression is equal to:

$$\begin{aligned} & \|J_{\Phi \circ R} \Lambda_{\Phi \circ R}(c \delta^{-1/2}) \underset{\nu^o}{\alpha \otimes \varepsilon} \Lambda_{\Phi \circ R}(d)\|^2 \\ &= (S_R(d^* d) J_{\Phi \circ R} \Lambda_{\Phi \circ R}(c \delta^{-1/2}) | J_{\Phi \circ R} \Lambda_{\Phi \circ R}(c \delta^{-1/2})) \\ &= (S_R(d^* d) \Lambda_{\Phi \circ R}(\sigma_{-i/2}^{\Phi \circ R}(\delta^{-1/2} c^*)) | \Lambda_{\Phi \circ R}(\sigma_{-i/2}^{\Phi \circ R}(\delta^{-1/2} c^*))) \end{aligned}$$

Then, properties of elements of C allow to finish the computation to get:

$$\begin{aligned} & \Phi \circ R((\omega_{J_{\Phi \circ R} \Lambda_{\Phi \circ R}(c)} \underset{\nu}{\beta \star \alpha} id)(\Gamma(d^* d))) \\ &= (S_R(d^* d) \delta^{-1/2} J_{\Phi \circ R} \Lambda_{\Phi \circ R}(c) | \delta^{-1/2} J_{\Phi \circ R} \Lambda_{\Phi \circ R}(c)) \end{aligned}$$

By continuity, the proposition holds. \square

COROLLARY 5.22. — *For all $v \in \mathcal{D}(\delta^{-1/2}) \cap D((H_{\Phi \circ R})_{\beta}, \nu^o)$ and for all element $a \in \mathcal{N}_{\Phi \circ R} \cap \mathcal{N}_{S_R}$, we have:*

$$S_R((\omega_v \underset{\nu}{\beta \star \alpha} id)(\Gamma(a))) = \alpha(< S_R(a) \delta^{-1/2} v, \delta^{-1/2} v >_{\beta, \nu^o})$$

Proof. — Straightforward by the formula $\nu \circ \alpha^{-1}(< x\xi, \eta >_{\beta, \nu^o}) = (x\xi | \eta)$ for all $x \in \beta(N)'$ and $\xi, \eta \in D(H_{\beta}, \nu^o)$, and the previous proposition. \square

We put $\Gamma^{(2)}$ the $*$ -homomorphism $(\Gamma \underset{N}{\beta \star \alpha} id) \circ \Gamma = (id \underset{N}{\beta \star \alpha} \Gamma) \circ \Gamma$ from M to $M \underset{N}{\beta \otimes \alpha} M \underset{N}{\beta \otimes \alpha} M$.

LEMMA 5.23. — *If $v \in \mathcal{D}(\delta^{-1/2} \underset{N}{\beta \otimes \alpha} \delta^{-1/2}) \cap D((H_{\Phi \circ R} \underset{\nu}{\beta \otimes \alpha} H_{\Phi \circ R})_{\beta}, \nu^o)$ and $a \in \mathcal{M}_{\Phi \circ R} \cap \mathcal{N}_{S_R}$, then we have:*

$$\begin{aligned} & \Phi \circ R((\omega_v \underset{\nu}{\beta \star \alpha} id)(\Gamma^{(2)}(a))) \\ &= ((S_R(a) \underset{N}{\beta \otimes \alpha} 1)(\delta^{-1/2} \underset{N}{\beta \otimes \alpha} \delta^{-1/2}) v | (\delta^{-1/2} \underset{N}{\beta \otimes \alpha} \delta^{-1/2}) v) \end{aligned}$$

Proof. — Let $\xi, \eta \in \mathcal{D}(\delta^{-1/2}) \cap D((H_{\Phi \circ R})_{\beta}, \nu^o)$. By the previous proposition and its corollary, we have:

$$\begin{aligned} & \Phi \circ R((\omega_{\xi \underset{\nu}{\beta \otimes \alpha} \eta} \underset{\nu}{\beta \star \alpha} id)(\Gamma^{(2)}(a))) \\ &= \Phi \circ R((\omega_{\eta \underset{\nu}{\beta \star \alpha} id})(\Gamma((\omega_{\xi \underset{\nu}{\beta \star \alpha} id})(\Gamma(a))))) \\ &= (S_R((\omega_{\xi \underset{\nu}{\beta \star \alpha} id})(\Gamma(a))) \delta^{-1/2} \eta | \delta^{-1/2} \eta) \\ &= (\alpha(< S_R(a) \delta^{-1/2} \xi, \delta^{-1/2} \xi >_{\beta, \nu^o} \delta^{-1/2} \eta | \delta^{-1/2} \eta)) \end{aligned}$$

which is equal to:

$$((S_R(a) \underset{N}{\beta \otimes \alpha} 1)(\delta^{-1/2} \xi \underset{\nu}{\beta \otimes \alpha} \delta^{-1/2} \eta) | \delta^{-1/2} \xi \underset{N}{\beta \otimes \alpha} \delta^{-1/2} \eta)$$

Since $\mathcal{D}(\delta^{-1/2}) \cap D((H_{\Phi \circ R})_{\beta}, \nu^o) \underset{\nu}{\beta \otimes \alpha} \mathcal{D}(\delta^{-1/2}) \cap D((H_{\Phi \circ R})_{\beta}, \nu^o)$ is a core for $\delta^{-1/2} \underset{N}{\beta \otimes \alpha} \delta^{-1/2}$, the lemma is proved. \square

LEMMA 5.24. — *We have:*

$$\Phi \circ R((\omega_v \underset{\nu}{\beta \star \alpha} id)(\Gamma^{(2)}(a))) = ((S_R(a) \underset{N}{\beta \otimes \alpha} 1)(\Gamma(\delta^{-1/2})v | (\Gamma(\delta^{-1/2})v))$$

for all $v \in \mathcal{D}(\Gamma(\delta^{-1/2})) \cap D((H_{\Phi \circ R} \underset{\nu}{\beta \otimes \alpha} H_{\Phi \circ R})_{\beta}, \nu^o)$ and $a \in \mathcal{N}_{\Phi \circ R} \cap \mathcal{N}_{S_R}$.

Proof. — Let $(\eta_i)_{i \in I}$ be a $D(H_{\beta}, \nu^o)$ -basis of H_{β} . We put $w_i = (\lambda_{\eta_i}^{\alpha, \hat{\beta}})^* Wv$ for all $i \in I$ and we have, for all $m \in M$:

$$\begin{aligned} (\Gamma(m)v | v) &= ((1 \underset{N^o}{\alpha \otimes \hat{\beta}} m)Wv | Wv) \\ &= \sum_{i \in I} ((\lambda_{\eta_i}^{\alpha, \hat{\beta}})^* (1 \underset{N^o}{\alpha \otimes \hat{\beta}} m)Wv | (\lambda_{\eta_i}^{\alpha, \hat{\beta}})^* Wv) = \sum_{i \in I} (mw_i | w_i) \end{aligned}$$

Since $v \in \mathcal{D}(\Gamma(\delta^{-1/2}))$ and $\Gamma(x) = W^*(1 \underset{N^o}{\alpha \otimes \hat{\beta}} x)W$, we notice that Wv belongs to $\mathcal{D}(1 \underset{N^o}{\alpha \otimes \hat{\beta}} \delta^{-1/2})$ and w_i belongs to $\mathcal{D}(\delta^{-1/2})$. Then, by the previous proposition and normality of $\Phi \circ R$, we have:

$$\begin{aligned} \Phi \circ R((\omega_v \underset{\nu}{\beta \star \alpha} id)(\Gamma^{(2)}(a))) &= \Phi \circ R((\omega_v \circ \Gamma \underset{\nu}{\beta \star \alpha} id)(\Gamma(a))) \\ &= \sum_{i \in I} \Phi \circ R((\omega_{w_i} \underset{\nu}{\beta \star \alpha} id)(\Gamma(a))) \\ &= \sum_{i \in I} (S_R(a) \delta^{-1/2} w_i | \delta^{-1/2} w_i) \\ &= ((S_R(a) \underset{N}{\beta \otimes \alpha} 1)(\Gamma(\delta^{-1/2})v | (\Gamma(\delta^{-1/2})v)) \end{aligned}$$

By continuity, the proposition holds. \square

PROPOSITION 5.25. — *We have $\Gamma(\delta) = \delta \underset{N}{\beta \otimes \alpha} \delta$.*

Proof. — For all element v in the intersection of $\mathcal{D}(\delta^{-1/2} \underset{N}{\beta \otimes \alpha} \delta^{-1/2})$, $\mathcal{D}(\Gamma(\delta^{-1/2}))$ and $D((H_{\Phi \circ R} \underset{\nu}{\beta \otimes \alpha} H_{\Phi \circ R})_{\beta}, \nu^o)$, we have:

$$\|(\delta^{-1/2} \underset{N}{\beta \otimes \alpha} \delta^{-1/2})v\|^2 = \|\Gamma(\delta^{-1/2})v\|^2$$

But, these operators commute each other so that there exists a subspace which is a core for both $\delta^{-1/2} \underset{N}{\beta \otimes_\alpha} \delta^{-1/2}$ and $\Gamma(\delta^{-1/2})$. By the previous equality, we get that the two operators have the same domain and consequently they are equal. \square

5.4. Uniqueness of left invariant operator-valued weight. —

THEOREM 5.26. — *If T' a n.s.f left invariant operator-valued weight which is β -adapted w.r.t ν , then there exists a strictly positive operator h affiliated with $Z(N)$ such that, for all $t \in \mathbb{R}$, we have:*

$$\nu \circ \alpha^{-1} \circ T' = (\nu \circ \alpha^{-1} \circ T_L)_{\beta(h)} \text{ and } [DT' : DT_L]_t = \beta(h^{it})$$

Proof. — We put $\Phi' = \nu \circ \alpha^{-1} \circ T'$. By 4.49, we have for all $s \in \mathbb{R}$:

$$\Gamma \circ \sigma_{-s}^\Phi \circ \sigma_s^{\Phi'} = (\tau_{-s} \underset{N}{\beta \star_\alpha} \sigma_{-s}^\Phi) \circ \Gamma \circ \sigma_s^{\Phi'} = (id \underset{N}{\beta \star_\alpha} \sigma_{-s}^\Phi \circ \sigma_s^{\Phi'}) \circ \Gamma$$

By right invariance T_R , we have for all $a \in \mathcal{M}_{T_R}^+$:

$$\begin{aligned} T_R(\sigma_{-s}^\Phi \circ \sigma_s^{\Phi'}(a)) &= (\Phi \circ R \underset{\nu}{\beta \star_\alpha} id)(\Gamma(\sigma_{-s}^\Phi \circ \sigma_s^{\Phi'}(a))) \\ &= \sigma_{-s}^\Phi \circ \sigma_s^{\Phi'}((\Phi \circ R \underset{\nu}{\beta \star_\alpha} id)\Gamma(a)) = \sigma_{-s}^\Phi \circ \sigma_s^{\Phi'}(T_R(a)) \end{aligned}$$

Since T and T' are α -adapted w.r.t ν , we get that $\Phi \circ R$ is $\sigma_{-s}^\Phi \circ \sigma_s^{\Phi'}$ -invariant and, so $\sigma_t^{\Phi \circ R}$ and $\sigma_{-s}^\Phi \circ \sigma_s^{\Phi'}$ commute each other. But $\sigma^{\Phi \circ R}$ and σ^Φ commute each other that's why $\sigma^{\Phi \circ R}$ and $\sigma^{\Phi'}$ also commute each other. For all $s, t \in \mathbb{R}$, we have:

$$\Gamma(\sigma_t^{\Phi'}(\delta^{is})\delta^{-is}) = (\tau_t \underset{N}{\beta \star_\alpha} \sigma_t^{\Phi'})(\Gamma(\delta^{is}))(\delta^{-is} \underset{N}{\beta \otimes_\alpha} \delta^{-is}) = 1 \underset{N}{\beta \otimes_\alpha} \sigma_t^{\Phi'}(\delta^{is})\delta^{-is}$$

Consequently $\sigma_t^{\Phi'}(\delta^{is})\delta^{-is}$ belongs to $\beta(N)$. Since T' is α -adapted w.r.t ν , $\beta(N)$ is $\sigma^{T'}$ -invariant and $M \cap \beta(N)'$ is $\sigma^{\Phi'}$ -invariant so that, in fact, $\sigma_t^{\Phi'}(\delta^{is})\delta^{-is}$ belongs to $\beta(Z(N))$ and we easily get that there exists a strictly positive operator k affiliated with $Z(N)$ such that $\sigma_t^{\Phi'}(\delta^{is}) = \beta(k^{ist})\delta^{is}$. Then, we have:

$$\begin{aligned} \sigma_s^{\Phi'} \circ \sigma_t^\Phi(m) &= \sigma_s^{\Phi'}(\delta^{-it} \sigma_t^{\Phi \circ R}(m) \delta^{it}) = \beta(k^{-ist}) \delta^{-it} \sigma_s^{\Phi'} \circ \sigma_t^{\Phi \circ R}(m) \delta^{it} \beta(k^{ist}) \\ &= \beta(k^{-ist}) \sigma_t^\Phi \circ \sigma_s^{\Phi'}(m) \beta(k^{ist}) \end{aligned}$$

Since T_L is β -adapted w.r.t ν , $\beta(k)$ is affiliated to the centralizer of σ^T . Apply Φ to the previous formula and get:

$$\begin{aligned} \Phi \circ \sigma_s^{\Phi'} \circ \sigma_t^\Phi(m^*m) &= \Phi(\beta(k^{-ist}) \sigma_t^\Phi \circ \sigma_s^{\Phi'}(m^*m) \beta(k^{ist})) \\ &= \Phi(\sigma_t^\Phi \circ \sigma_s^{\Phi'}(m^*m)) = \Phi \circ \sigma_s^{\Phi'}(m^*m) \end{aligned}$$

So, by 5.9 and left invariance $\sigma_{-s}^{\Phi'} \circ T_L \circ \sigma_s^{\Phi'}$, there exists a strictly positive operator q_s affiliated with $Z(N)$ such that $\Phi \circ \sigma_s^{\Phi'} = \Phi_{\beta(q_s)}$. By usual arguments, we deduce that there exists a strictly positive q affiliated to $Z(N)$ such that $\Phi \circ \sigma_s^{\Phi'} = \Phi_{\beta(q^{-s})}$ and $[D\Phi \circ \sigma_s^{\Phi'} : D\Phi]_s = \beta(q^{-s})$. Then, again by 5.9, there exists a strictly positive operator h affiliated to $Z(N)$ such that $\Phi = \Phi_{\beta(h)}$ avec $[DT' : DT_L]_t = \beta(h^{it})$. \square

Also, we have a similar result for right invariant operator-valued weight.

COROLLARY 5.27. — *There exists a strictly positive operator h affiliated with $Z(N)$ such that:*

$$T_R = (R \circ T_L \circ R)_{\alpha(h)}$$

Proof. — T_R and $R \circ T_L \circ R$ satisfy hypothesis of the previous theorem. \square

5.5. Manageability of the fundamental unitary. — In this section, we prove that the fundamental unitary satisfies a proposition similar to Woronowicz's manageability of [Wor96].

LEMMA 5.28. — *There exists a strictly positive operator P on H_Φ such that, for all $x \in \mathcal{N}_\Phi$ and $t \in \mathbb{R}$, we have $P^{it} \Lambda_\Phi(x) = \lambda^{\frac{t}{2}} \Lambda_\Phi(\tau_t(x))$.*

Proof. — Since $\Phi \circ R = \Phi_\delta$, by [Vae01a] (5.3), we have:

$$\Lambda_\Phi(\sigma_t^{\Phi \circ R}(x)) = \delta^{it} J_\Phi \lambda^{\frac{t}{2}} \delta^{it} J_\Phi \Delta_\Phi^{it} \Lambda_\Phi(x)$$

and since λ is affiliated with $Z(M)$, we get $\|\Lambda_\Phi(\sigma_t^{\Phi \circ R}(x))\| = \|\lambda^{\frac{t}{2}} \Lambda_\Phi(x)\|$ for all $x \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$ and $t \in \mathbb{R}$. But, we know that Φ is $\sigma_t^{\Phi \circ R} \circ \tau_t$ -invariant, so $\|\Lambda_\Phi(x)\| = \|\lambda^{\frac{t}{2}} \Lambda_\Phi(\tau_t(x))\|$. Then, there exists P_t on H_Φ such that:

$$P_t \Lambda_\Phi(x) = \lambda^{\frac{t}{2}} \Lambda_\Phi(\tau_t(x))$$

for all $x \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$ and $t \in \mathbb{R}$. For all $s, t \in \mathbb{R}$, we verify that $P_s P_t = P_{st}$ thanks to relation $\tau_t(\lambda) = \lambda$ and the existence of P follows. \square

DEFINITION 5.29. — We call **manageable operator** the strictly positive operator P on H_Φ such that $P^{it} \Lambda_\Phi(x) = \lambda^{\frac{t}{2}} \Lambda_\Phi(\tau_t(x))$, for all $x \in \mathcal{N}_\Phi$ and $t \in \mathbb{R}$.

PROPOSITION 5.30. — *For all $m \in M$, $n \in N$ and $t \in \mathbb{R}$, we have:*

$$\begin{aligned} P^{it} m P^{-it} &= \tau_t(m) & P^{it} \alpha(n) P^{-it} &= \alpha(\sigma_t^\nu(n)) \\ P^{it} \beta(n) P^{-it} &= \beta(\sigma_t^\nu(n)) & P^{it} \hat{\beta}(n) P^{-it} &= \hat{\beta}(\sigma_t^\nu(n)) \end{aligned}$$

Proof. — Straightforward. \square

Then, we can define operators $P^{it} \underset{\nu}{\beta \otimes \alpha} P^{it}$ on $H_\Phi \underset{\nu}{\beta \otimes \alpha} H_\Phi$ and $P^{it} \underset{\nu^\circ}{\alpha \otimes \hat{\beta}} P^{it}$ on $H_\Phi \underset{\nu^\circ}{\alpha \otimes \hat{\beta}} H_\Phi$ for all $t \in \mathbb{R}$.

THEOREM 5.31. — *W satisfies a manageability relation. More exactly, we have:*

$$\begin{aligned} & (\sigma_\nu W^* \sigma_\nu (q \underset{\nu}{\beta} \otimes_\alpha v) | p \underset{\nu^o}{\alpha} \otimes_\beta w) \\ &= (\sigma_{\nu^o} W \sigma_{\nu^o} (J_\Phi p \underset{\nu^o}{\alpha} \otimes_\beta P^{-1/2} v) | J_\Phi q \underset{\nu}{\beta} \otimes_\alpha P^{1/2} w) \end{aligned}$$

for all $v \in \mathcal{D}(P^{-\frac{1}{2}})$, $w \in \mathcal{D}(P^{\frac{1}{2}})$ and $p, q \in D({}_\alpha H_\Phi, \nu) \cap D((H_\Phi)_\beta, \nu^o)$. Moreover, we have $W(P^{it} \underset{\nu}{\beta} \otimes_\alpha P^{it}) = (P^{it} \underset{\nu^o}{\alpha} \otimes_\beta P^{it})W$ for all $t \in \mathbb{R}$.

Proof. — Let $p, q \in D({}_\alpha H_\Phi, \nu) \cap D((H_\Phi)_\beta, \nu^o)$. For all $v \in \mathcal{D}(D^{1/2})$ and $w \in \mathcal{D}(D^{-1/2})$, we know that:

$$(I(id * \omega_{q,p})(W)Iv|w) = ((id * \omega_{p,q})(W)D^{1/2}v|D^{-1/2}w)$$

Since $(id * \omega_{p,q})(W) \in \mathcal{D}(S) = \mathcal{D}(\tau_{-i/2})$ and τ is implemented by D^{-1} , we have $\tau_{-i/2}((id * \omega_{p,q})(W)) = I(id * \omega_{q,p})(W)I$. But τ is also implemented by P , so that:

$$(I(id * \omega_{q,p})(W)Iv|w) = ((id * \omega_{p,q})(W)P^{1/2}v|P^{-1/2}w)$$

for all $v \in \mathcal{D}(P^{1/2})$ and $w \in \mathcal{D}(P^{-1/2})$. By 4.52, we rewrite the formula:

$$\begin{aligned} & (\sigma_\nu W^* \sigma_\nu (q \underset{\nu}{\beta} \otimes_\alpha v) | p \underset{\nu^o}{\alpha} \otimes_\beta w) \\ &= (\sigma_{\nu^o} W \sigma_{\nu^o} (J_\Phi p \underset{\nu^o}{\alpha} \otimes_\beta P^{-1/2} v) | J_\Phi q \underset{\nu}{\beta} \otimes_\alpha P^{1/2} w) \end{aligned}$$

Now, we have to prove $W^*(P^{it} \underset{\nu^o}{\alpha} \otimes_\beta P^{it}) = (P^{it} \underset{\nu}{\beta} \otimes_\alpha P^{it})W^*$ for all $t \in \mathbb{R}$. First of all, because of the commutation relation between P and β , $D((H_\Phi)_\beta, \nu^o)$ is P^{it} -invariant and if $(\xi_i)_{i \in I}$ is a (N^o, ν^o) -basis of $(H_\Phi)_\beta$, then $(P^{it}\xi_i)_{i \in I}$ is also. Let $v \in D((H_\Phi)_\beta, \nu^o)$ and $a \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$. We compute:

$$\begin{aligned} & (P^{it} \underset{\nu}{\beta} \otimes_\alpha P^{it})W^*(v \underset{\nu^o}{\alpha} \otimes_\beta \Lambda_\Phi(a)) \\ &= \sum_{i \in I} P^{it}\xi_i \underset{\nu}{\beta} \otimes_\alpha \lambda^{t/2} \Lambda_\Phi(\tau_t((\omega_{v, \xi_i} \underset{\nu}{\beta} \star_\alpha id)(\Gamma(a)))) \\ &= \sum_{i \in I} P^{it}\xi_i \underset{\nu}{\beta} \otimes_\alpha \Lambda_\Phi((\omega_{P^{it}v, P^{it}\xi_i} \underset{\nu}{\beta} \star_\alpha id)(\Gamma(\lambda^{t/2}\tau_t(a)))) \\ &= W^*(P^{it}v \underset{\nu^o}{\alpha} \otimes_\beta \lambda^{t/2} \Lambda_\Phi(\tau_t(a))) = W^*(P^{it} \underset{\nu^o}{\alpha} \otimes_\beta P^{it})(v \underset{\nu^o}{\alpha} \otimes_\beta \Lambda_\Phi(a)) \end{aligned}$$

□

Following [Eno02] (definition 4.1), we define the notion of weakly regular pseudo-multiplicative unitary.

DEFINITION 5.32. — A pseudo-multiplicative unitary \mathcal{W} w.r.t $\alpha, \beta, \hat{\beta}$ is said to be **weakly regular** if the weakly closed linear span of $(\lambda_v^{\alpha, \beta})^* \mathcal{W} \rho_w^{\hat{\beta}, \alpha}$ where v, w belongs to $D(\alpha H, \nu)$ is equal to $\alpha(N)'$.

PROPOSITION 5.33. — The operator $\widehat{W} = \sigma_\nu W^* \sigma_\nu$ from $H_{\Phi} \underset{\nu}{\hat{\beta} \otimes \alpha} H_{\Phi}$ to $H_{\Phi} \underset{\nu^o}{\alpha \otimes \beta} H_{\Phi}$ is a pseudo-multiplicative unitary over N w.r.t $\alpha, \beta, \hat{\beta}$ which is weakly regular in the sense of [Eno02] (definition 4.1).

Proof. — By [EV00], we know that \widehat{W} is a pseudo-multiplicative unitary. We also know that $\langle (\lambda_v^{\alpha, \beta})^* \widehat{W} \rho_w^{\hat{\beta}, \alpha} \rangle^{-w} \subset \alpha(N)'$. For all $v \in \mathcal{D}(P^{-\frac{1}{2}})$, $w \in \mathcal{D}(P^{\frac{1}{2}})$ and $p, q \in D(\alpha H_{\Phi}, \nu) \cap D((H_{\Phi})_{\hat{\beta}}, \nu^o)$, we have, by theorem 5.31:

$$((\lambda_p^{\alpha, \beta})^* \widehat{W} \rho_v^{\hat{\beta}, \alpha} q | w) = (\sigma_{\nu^o} W \sigma_{\nu^o} (J_{\Phi} p \underset{\nu^o}{\alpha \otimes \beta} P^{-1/2} v) | J_{\Phi} q \underset{\nu}{\hat{\beta} \otimes \alpha} P^{1/2} w)$$

and on the other hand:

$$\begin{aligned} (R^{\alpha, \nu}(v) R^{\alpha, \nu}(p)^* q | w) &= (R^{\alpha, \nu}(v) J_{\nu} R^{\hat{\beta}, \nu^o} (J_{\Phi} p)^* J_{\Phi} q | w) \\ &= (R^{\alpha, \nu}(v) J_{\nu} \Lambda_{\nu} (\langle J_{\Phi} q, J_{\Phi} p \rangle_{\hat{\beta}, \nu_L^o}) | w) \\ &= (P^{-1/2} R^{\alpha, \nu}(v) J_{\nu} \Lambda_{\nu} (\langle J_{\Phi} q, J_{\Phi} p \rangle_{\hat{\beta}, \nu_L^o}) | P^{1/2} w) \\ &= (R^{\alpha, \nu}(P^{-1/2} v) \Delta_{\nu}^{-1/2} J_{\nu} \Lambda_{\nu} (\langle J_{\Phi} q, J_{\Phi} p \rangle_{\hat{\beta}, \nu_L^o}) | P^{1/2} w) \\ &= (R^{\alpha, \nu}(P^{-1/2} v) \Lambda_{\nu} (\langle J_{\Phi} p, J_{\Phi} q \rangle_{\hat{\beta}, \nu_L^o}) | P^{1/2} w) \\ &= (\alpha(\langle J_{\Phi} p, J_{\Phi} q \rangle_{\hat{\beta}, \nu_L^o}) P^{-1/2} v | P^{1/2} w) \\ &= (J_{\Phi} p \underset{\nu}{\hat{\beta} \otimes \alpha} P^{-1/2} v | J_{\Phi} q \underset{\nu}{\hat{\beta} \otimes \alpha} P^{1/2} w) \end{aligned}$$

There exists $\Xi \in H_{\Phi} \underset{\nu}{\hat{\beta} \otimes \alpha} H_{\Phi}$ such that $\sigma_{\nu^o} W \sigma_{\nu^o} \Xi = J_{\Phi} p \underset{\nu}{\hat{\beta} \otimes \alpha} P^{-1/2} v$ since W is onto. By definition, there exists a net $(\sum_{k=1}^{n(i)} J_{\Phi} p_k^i \underset{\nu^o}{\alpha \otimes \beta} P^{-1/2} v_k^i)_{i \in I}$ which converges to Ξ . So, the net $((\sum_{k=1}^{n(i)} (\lambda_{p_k^i}^{\alpha, \beta})^* \widehat{W} \rho_{v_k^i}^{\hat{\beta}, \alpha} q | w))_{i \in I}$ converges to:

$$\begin{aligned} (\sigma_{\nu^o} W \sigma_{\nu^o} \Xi | J_{\Phi} q \underset{\nu}{\hat{\beta} \otimes \alpha} P^{1/2} w) &= (J_{\Phi} p \underset{\nu}{\hat{\beta} \otimes \alpha} P^{-1/2} v | J_{\Phi} q \underset{\nu}{\hat{\beta} \otimes \alpha} P^{1/2} w) \\ &= (R^{\alpha, \nu}(v) R^{\alpha, \nu}(p)^* q | w) \end{aligned}$$

Then, we obtain $\alpha(N)' = \langle R^{\alpha, \nu}(v) R^{\alpha, \nu}(p)^* \rangle^{-w} \subset \langle (\omega_{v, p} * id)(\widehat{W} \sigma_{\nu^o}) \rangle^{-w}$. \square

COROLLARY 5.34. — If \hat{M} denote the weak closed linear span of $(\omega_{\xi, \eta} * id)(W)$ where $\xi \in D((H_{\Phi})_{\hat{\beta}}, \nu^o)$ and $\eta \in D(\alpha H_{\Phi}, \nu)$, then \hat{M} is a von Neumann algebra.

Proof. — Comes from weak regularity of \widehat{W} and [Eno02] (proposition 3.2). \square

5.6. Changing the quasi-invariant weight. — Let ν' be a n.s.f weight on N such that there exist strictly positive operator h and k affiliated with N strongly commuting and $[D\nu' : D\nu]_t = k^{\frac{it^2}{2}} h^{it}$ for all $t \in \mathbb{R}$. By [Vae01a] (proposition 5.1), it is equivalent to $\sigma_t^{\nu'}(h^{is}) = k^{ist} h^{is}$ for all $s, t \in \mathbb{R}$ and $\nu' = \nu_h$ in the sense of [Vae01a]. This hypothesis is satisfied, in particular, if σ^ν and $\sigma^{\nu'}$ commute each other. In this cas, k is affiliated with $Z(N)$.

PROPOSITION 5.35. — *There exists a n.s.f operator-valued weight T'_L from M to $\alpha(N)$ which is β -adapted w.r.t ν' such that, for all $t \in \mathbb{R}$, we have:*

$$[DT'_L : DT_L]_t = \beta(k^{\frac{-it^2}{2}} h^{it})$$

Proof. — By 4.4, there exists a n.s.f operator-valued weight S_L from M to $\beta(N)$ such that $\nu \circ \alpha^{-1} \circ T_L = \nu \circ \beta^{-1} \circ S_L$ so that S_L is α -adapted w.r.t ν . Then, again by 4.4, there exists a n.s.f operator-valued weight T'_L from M to $\alpha(N)$ such that $\nu' \circ \beta^{-1} \circ S = \nu \circ \alpha^{-1} \circ T'_L$ so that T'_L is β -adapted w.r.t ν' . Then, we compute the Radon-Nikodym cocycle for all $t \in \mathbb{R}$:

$$\begin{aligned} [DT'_L : DT_L]_t &= [D\nu \circ \alpha^{-1} \circ T'_L : D\nu \circ \alpha^{-1} \circ T_L]_t \\ &= [D\nu' \circ \beta^{-1} \circ S : D\nu \circ \beta^{-1} \circ S]_t \\ &= \beta([D\nu' : D\nu]_{-t}^*) = \beta(k^{\frac{-it^2}{2}} h^{it}) \end{aligned}$$

□

COROLLARY 5.36. — *We have:*

$$\nu \circ \alpha^{-1} \circ T'_L = (\nu \circ \alpha^{-1} \circ T_L)_{\beta(h)} \quad \text{and} \quad \nu' \circ \alpha^{-1} \circ T'_L = (\nu \circ \alpha^{-1} \circ T_L)_{\alpha(h)\beta(h)}$$

Proof. — Come from [Vae01a] (proposition 5.1) and the following equality, for all $t \in \mathbb{R}$, $[D\nu' \circ \alpha^{-1} \circ T'_L : D\nu \circ \alpha^{-1} \circ T_L]_t = \alpha(k^{\frac{it^2}{2}}) \beta(k^{\frac{-it^2}{2}}) \alpha(h^{it}) \beta(h^{it})$. □

PROPOSITION 5.37. — *T'_L is left invariant.*

Proof. — Let $a \in \mathcal{M}_{T'_L}^+$. By left invariance of T_L , we have:

$$\begin{aligned} (id_{\beta \star_{\alpha} \nu'} \nu' \circ \alpha^{-1} \circ T'_L)(\Gamma(a)) &= (id_{\beta \star_{\alpha} \nu} \nu \circ \alpha^{-1} \circ T'_L)(\Gamma(a)) \\ &= (id_{\beta \star_{\alpha} \nu} (\nu \circ \alpha^{-1} \circ T_L)_{\beta(h)})(\Gamma(a)) \\ &= (id_{\beta \star_{\alpha} \nu} \nu \circ \alpha^{-1} \circ T_L)(\Gamma(\beta(h^{1/2}) a \beta(h^{1/2}))) \\ &= T_L(\beta(h^{1/2}) a \beta(h^{1/2})) = T'(a) \end{aligned}$$

□

We state the right version of these results:

PROPOSITION 5.38. — *There exists a n.s.f right invariant operator-valued weight T'_R which is α -adapted w.r.t ν' such that, for all $t \in \mathbb{R}$, we have:*

$$[DT'_R : DT_R]_t = \alpha(k^{\frac{it^2}{2}} h^{it})$$

Moreover, we have:

$$\nu \circ \beta^{-1} \circ T'_R = (\nu \circ \beta^{-1} \circ T_R)_{\alpha(h)} \quad \text{and} \quad \nu' \circ \beta^{-1} \circ T'_R = (\nu \circ \beta^{-1} \circ T_R)_{\alpha(h)\beta(h)}$$

LEMMA 5.39. — *The application $I_{\nu'}^{\nu'}$ defined by the following formula:*

$$I_{\nu'}^{\nu'}(\xi_{\beta \otimes_{\alpha} \eta}) = \beta(k^{-i/8}) \xi_{\beta \otimes_{\alpha} \alpha(h^{1/2}) \eta}$$

for all $\xi \in H$ and $\eta \in D(\alpha H, \nu) \cap \mathcal{D}(\alpha(h^{1/2}))$, is an isomorphism of $\beta(N)' - \alpha(N)'^o$ -bimodules from $H_{\beta \otimes_{\alpha} H}$ onto $H_{\beta \otimes_{\alpha} H}$.

Proof. — For all $x \in \mathcal{N}_{\nu'}$, we have:

$$\alpha(x) \alpha(h^{1/2}) \eta = \alpha(x h^{1/2}) \eta = R^{\alpha, \nu}(\eta) \Lambda_{\nu}(x h^{1/2}) = R^{\alpha, \nu}(\eta) \Lambda_{\nu'}(x)$$

so that $\alpha(h^{1/2}) \eta \in D(\alpha H, \nu)$ and $R^{\alpha, \nu'}(\alpha(h^{1/2}) \eta) = R^{\alpha, \nu}(\eta)$. Also, we recall that $J_{\nu'} = J_{\nu} k^{-i/8} J_{\nu} k^{i/8} J_{\nu}$ by [Vae01a] (proposition 2.5). Then, we have:

$$\begin{aligned} & (\beta(k^{-i/8}) \xi_1_{\beta \otimes_{\alpha} \alpha(h^{1/2}) \eta_1} | \beta(k^{-i/8}) \xi_2_{\beta \otimes_{\alpha} \alpha(h^{1/2}) \eta_2}) \\ &= (\beta(J_{\nu'} < \alpha(h^{1/2}) \eta_1, \alpha(h^{1/2}) \eta_2 >_{\alpha, \nu'}^* J_{\nu'}) \beta(k^{-i/8}) \xi_1 | \beta(k^{-i/8}) \xi_2) \\ &= (\beta(k^{-i/8} J_{\nu} k^{-i/8} J_{\nu} k^{i/8} J_{\nu} < \eta_1, \eta_2 >_{\alpha, \nu}^* J_{\nu} k^{-i/8} J_{\nu} k^{i/8} J_{\nu} k^{i/8}) \xi_1 | \xi_2) \\ &= (\beta(J_{\nu} < \eta_1, \eta_2 >_{\alpha, \nu}^* J_{\nu}) \xi_1 | \xi_2) = (\xi_1_{\beta \otimes_{\alpha} \eta_1} | \xi_2_{\beta \otimes_{\alpha} \eta_2}) \end{aligned}$$

□

REMARK 5.40. — For all $\xi \in D(H_{\beta}, \nu^o)$ and $\eta \in D(\alpha H, \nu)$, we have:

$$\begin{aligned} I_{\nu'}^{\nu'}(\xi_{\beta \otimes_{\alpha} \eta}) &= \beta(k^{-i/8}) \xi_{\beta \otimes_{\alpha} \alpha(h^{1/2}) \eta} = \xi_{\beta \otimes_{\alpha} \alpha(k^{-i/8} h^{1/2}) \eta} \\ &= \beta(\sigma_{i/2}^{\nu}(h^{1/2})) \xi_{\beta \otimes_{\alpha} \alpha(k^{-i/8}) \eta} = \beta(\sigma_{i/2}^{\nu}(k^{-i/8} h^{1/2})) \xi_{\beta \otimes_{\alpha} \eta} \\ &= \beta(k^{i/8}) \xi_{\beta \otimes_{\alpha} \alpha(\sigma_{-i/2}^{\nu}(h^{1/2})) \eta} = \beta(k^{i/8} h^{1/2}) \xi_{\beta \otimes_{\alpha} \eta} \end{aligned}$$

PROPOSITION 5.41. — *Let $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$ be a measured quantum groupoid. There exists a measured quantum groupoid $(N, M, \alpha, \beta, \Gamma, \nu', T'_L, T'_R)$ fundamental objects of which, $R', \tau', \lambda', \delta'$ and P' , are expressed, for all $t \in \mathbb{R}$, in the following way:*

$$\begin{aligned}
i) \quad & R' = R, \lambda' = \lambda \text{ and } \delta' = \delta; \\
ii) \quad & \tau'_t = Ad_{\alpha(k^{\frac{-it^2}{2}}h^{-it})\beta(k^{\frac{it^2}{2}}h^{it})} \circ \tau_t = Ad_{\alpha([D\nu':D\nu]_t^*)\beta([D\nu':D\nu]_t)} \circ \tau_t \\
iii) \quad & P'^{it} = \alpha(k^{\frac{it^2}{2}}h^{it})\beta(k^{\frac{-it^2}{2}}h^{-it})J_\Phi\alpha(k^{\frac{it^2}{2}}h^{it})\beta(k^{\frac{-it^2}{2}}h^{-it})J_\Phi P^{it}
\end{aligned}$$

Proof. — The existence of $(N, M, \alpha, \beta, \Gamma, \nu', T'_L, T'_R)$ has been already proved. We put $\Phi' = \nu' \circ \alpha^{-1} \circ T'_L$ and $\Psi' = \nu' \circ \beta^{-1} \circ T'_R$. Let $x, y \in \mathcal{N}_{T'_R} \cap \mathcal{N}_{\Psi'}$. By [Vae01a] (proposition 2.5), we have:

$$\begin{aligned}
J_{\Psi'}\Lambda_{\Psi'}(x) &= J_{\Psi}\alpha(k^{-i/8})\beta(k^{i/8})J_{\Psi}\alpha(k^{i/8})\beta(k^{-i/8})J_{\Psi}\Lambda_{\Psi}(x\alpha(h^{1/2})\beta(h^{1/2})) \\
\omega_{J_{\Psi'}\Lambda_{\Psi'}(x)} &= \omega_{\alpha(k^{i/8})\beta(k^{-i/8})J_{\Psi}\Lambda_{\Psi}(x\alpha(h^{1/2})\beta(h^{1/2}))}
\end{aligned}$$

Then, we easily verify

$$\lambda_{\alpha(k^{i/8})\beta(k^{-i/8})J_{\Psi}\Lambda_{\Psi}(x\alpha(h^{1/2})\beta(h^{1/2}))}^{\beta, \alpha, \nu'} = I_{\nu'}^{\nu'} \lambda_{\alpha(k^{i/8})J_{\Psi}\Lambda_{\Psi}(x\alpha(h^{1/2}))}^{\beta, \alpha, \nu}$$

We compute:

$$\begin{aligned}
& (\omega_{J_{\Psi'}\Lambda_{\Psi'}(x)} \underset{\nu'}{\beta \star \alpha} id)(\Gamma(y^*y)) \\
&= (\omega_{\alpha(k^{i/8})\beta(k^{-i/8})J_{\Psi}\Lambda_{\Psi}(x\alpha(h^{1/2})\beta(h^{1/2}))} \underset{\nu'}{\beta \star \alpha} id)(\Gamma(y^*y)) \\
&= (\omega_{\alpha(k^{i/8})J_{\Psi}\Lambda_{\Psi}(x\alpha(h^{1/2}))} \underset{\nu}{\beta \star \alpha} id)(\Gamma(y^*y)) \\
&= (\omega_{J_{\Psi}\Lambda_{\Psi}(x\alpha(k^{-i/8}h^{1/2}))} \underset{\nu}{\beta \star \alpha} id)(\Gamma(y^*y))
\end{aligned}$$

Apply R to get:

$$\begin{aligned}
& R[(\omega_{J_{\Psi'}\Lambda_{\Psi'}(x)} \underset{\nu'}{\beta \star \alpha} id)(\Gamma(y^*y))] \\
&= (\omega_{J_{\Psi}\Lambda_{\Psi}(y)} \underset{\nu}{\beta \star \alpha} id)(\Gamma(\alpha(k^{i/8}h^{1/2})x^*x\alpha(k^{-i/8}h^{1/2}))) \\
&= (\omega_{\alpha(k^{-i/8}h^{1/2})J_{\Psi}\Lambda_{\Psi}(y)} \underset{\nu}{\beta \star \alpha} id)(\Gamma(x^*x)) \\
&= (\omega_{\alpha(k^{i/8})J_{\Psi}\Lambda_{\Psi}(y\alpha(h^{1/2}))} \underset{\nu}{\beta \star \alpha} id)(\Gamma(x^*x)) \\
&= (\omega_{J_{\Psi'}\Lambda_{\Psi'}(y)} \underset{\nu'}{\beta \star \alpha} id)(\Gamma(x^*x)) = R'[(\omega_{J_{\Psi'}\Lambda_{\Psi'}(x)} \underset{\nu'}{\beta \star \alpha} id)(\Gamma(y^*y))]
\end{aligned}$$

so that $R = R'$. For all $a \in M$, $\xi \in D(H_\beta, \nu'^o)$ and $t \in \mathbb{R}$, we have:

$$\begin{aligned}
& \tau_t((\omega_\xi \underset{\nu'}{\beta \star \alpha} id)(\Gamma(a))) \\
&= \tau_t(\alpha(k^{-i/8}h^{-1/2})(\omega_\xi \underset{\nu}{\beta \star \alpha} id)(\Gamma(a))\alpha(k^{i/8}h^{-1/2})) \\
&= \alpha(\sigma_t^\nu(k^{-i/8}h^{-1/2}))\tau_t((\omega_\xi \underset{\nu}{\beta \star \alpha} id)(\Gamma(a))\alpha(\sigma_t^\nu(k^{i/8}h^{-1/2}))) \\
&= \alpha(k^{-t/2-i/8}h^{-1/2})(\omega_{\Delta_\Psi^{-it}\xi} \underset{\nu}{\beta \star \alpha} id)(\Gamma(\sigma_{-t}^\Psi(a)))\alpha(k^{-t/2+i/8}h^{-1/2})
\end{aligned}$$

By [Vae01a] (proposition 2.4 and corollaire 2.6), we know that:

$$\begin{aligned} & (\omega_{\Delta_{\Psi}^{-it}\xi} \beta \star_{\alpha} id)(\Gamma(\sigma_{-t}^{\Psi}(a))) \\ &= (\omega_{\alpha(k\frac{-it^2}{2}h^{it})\beta(k\frac{it^2}{2}h^{it})\Delta_{\Psi'}^{-it}\xi} \beta \star_{\alpha} id)(\Gamma(Ad_{\alpha(k\frac{it^2}{2}h^{-it})\beta(k\frac{-it^2}{2}h^{-it})} \circ \sigma_{-t}^{\Psi'}(a))) \end{aligned}$$

so that:

$$\begin{aligned} & \tau_t((\omega_{\xi} \beta \star_{\alpha} id)(\Gamma(a))) \\ &= Ad_{\alpha(k^{-t/2+i/8}h^{-1/2})\beta(k\frac{it^2}{2}h^{it})} \circ (\omega_{\beta(k\frac{it^2}{2}h^{it})\Delta_{\Psi'}^{-it}\xi} \beta \star_{\alpha} id)(\Gamma(\sigma_{-t}^{\Psi'}(a))) \\ &= \alpha(k\frac{-it^2}{2}h^{-it})\beta(k\frac{it^2}{2}h^{it})(\omega_{\Delta_{\Psi'}^{-it}\xi} \beta \star_{\alpha} id)(\Gamma(\sigma_{-t}^{\Psi'}(a)))\alpha(k\frac{it^2}{2}h^{it})\beta(k\frac{-it^2}{2}h^{-it}) \\ &= \alpha(k\frac{-it^2}{2}h^{-it})\beta(k\frac{it^2}{2}h^{it})\tau'_t((\omega_{\xi} \beta \star_{\alpha} id)(\Gamma(a)))\alpha(k\frac{it^2}{2}h^{it})\beta(k\frac{-it^2}{2}h^{-it}) \end{aligned}$$

Consequently, we have:

$$\tau'_t(z) = \alpha(k\frac{it^2}{2}h^{it})\beta(k\frac{-it^2}{2}h^{-it})\tau_t(z)\alpha(k\frac{-it^2}{2}h^{-it})\beta(k\frac{it^2}{2}h^{it})$$

for all $z \in M$ and $t \in \mathbb{R}$. Now, we compute the Radon-Nikodym cocycle:

$$\begin{aligned} & [D\nu' \circ \alpha^{-1} \circ T' \circ R : D\nu' \circ \alpha^{-1} \circ T']_t \\ &= [D\nu' \alpha^{-1} T' R : D\nu \alpha^{-1} T R]_t [D\nu \alpha^{-1} T R : D\nu \alpha^{-1} T]_t [D\nu \alpha^{-1} T : D\nu' \alpha^{-1} T']_t \\ &= \alpha([D\nu' : D\nu]_t) \beta([D\nu' : D\nu]_{-t}^*) \lambda^{\frac{it^2}{2}} \delta^{it} \alpha([D\nu : D\nu']_t) \beta([D\nu : D\nu']_{-t}^*) \end{aligned}$$

which is equal to $\lambda^{\frac{it^2}{2}} \delta^{it}$. Finally, we express the manageable operator P' in terms of P . We have, for all $x \in \mathcal{N}_{T'_L} \cap \mathcal{N}_{\Phi'}$ and $t \in \mathbb{R}$:

$$\begin{aligned} & P'^{it} \Lambda_{\Phi'}(x) = \lambda^{t/2} \Lambda_{\Phi'}(\tau'_t(x)) \\ &= \lambda^{t/2} \Lambda_{\Phi}(\alpha(k\frac{it^2}{2}h^{it})\beta(k\frac{-it^2}{2}h^{-it})\tau_t(x)\alpha(k\frac{-it^2}{2}h^{-it})\beta(k\frac{it^2}{2}h^{it})\alpha(h^{1/2})\beta(h^{1/2})) \end{aligned}$$

which is equal to the value of:

$$\lambda^{t/2} \alpha(k\frac{it^2}{2}h^{it})\beta(k\frac{-it^2}{2}h^{-it})J_{\Phi}\alpha(k\frac{it^2}{2}h^{it})\beta(k\frac{-it^2}{2}h^{-it})\alpha(k^{t/2})\beta(k^{t/2})J_{\Phi}$$

on $\Lambda_{\Phi}(\tau_t(x)\alpha(h^{1/2})\beta(h^{1/2}))$ and the value of:

$$\lambda^{t/2} \alpha(k\frac{it^2}{2}h^{it})\beta(k\frac{-it^2}{2}h^{-it})J_{\Phi}\alpha(k\frac{it^2}{2}h^{it})\beta(k\frac{-it^2}{2}h^{-it})J_{\Phi}$$

on $\Lambda_{\Phi}(\tau_t(x)\alpha(h^{1/2})\beta(h^{1/2}))$ which is:

$$\alpha(k\frac{it^2}{2}h^{it})\beta(k\frac{-it^2}{2}h^{-it})J_{\Phi}\alpha(k\frac{it^2}{2}h^{it})\beta(k\frac{-it^2}{2}h^{-it})J_{\Phi}P^{it}\Lambda_{\Phi'}(x)$$

□

Thanks to these formulas, we verify for example that $\tau'_t(\alpha(n)) = \alpha(\sigma_t^{\nu'}(n))$, $\tau'_t(\beta(n)) = \beta(\sigma_t^{\nu'}(n))$ and τ' is implemented by P' .

PROPOSITION 5.42. — *Let $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$ be measured quantum groupoid and let \tilde{T}_L be an other n.s.f left invariant operator-valued weight which is β -adapted w.r.t ν . Then fundamental objects \tilde{R} , $\tilde{\tau}$, $\tilde{\lambda}$, $\tilde{\delta}$ and \tilde{P} of the measured quantum groupoid $(N, M, \alpha, \beta, \Gamma, \nu, \tilde{T}_L, T_R)$ can be expressed in the following way:*

- i) $\tilde{R} = R$, $\tilde{\tau} = \tau$, $\tilde{\lambda} = \lambda$ and $\tilde{P} = P$
- ii) $\tilde{\delta} = \delta\alpha(h)\beta(h^{-1})$ where h is affiliated with $Z(N)$ s.t. $\tilde{T}_L = (T_L)_{\beta(h)}$

Proof. — By uniqueness theorem, there exists a strictly positive operator h affiliated with $Z(N)$ such that $\nu \circ \alpha^{-1} \circ \tilde{T}_L = (\nu \circ \alpha^{-1} \circ T_L)_{\beta(h)}$ and, for all $t \in \mathbb{R}$, we have $[D\tilde{T}_L : DT_L]_t = \beta(h^{it})$. We have already noticed that R and τ are independent w.r.t left invariant operator-valued weight and β -adapted w.r.t ν . We compute then Radon-Nydodim cocycle:

$$\begin{aligned} & [D\nu\beta^{-1}R\tilde{T}_LR : D\nu\alpha^{-1}\tilde{T}_L]_t \\ &= [D\nu\beta^{-1}R\tilde{T}_LR : D\nu\beta^{-1}RT_LR]_t [D\Psi : D\Phi]_t [D\nu\alpha^{-1}T_L : D\nu\alpha^{-1}\tilde{T}_L]_t \\ &= R([D\tilde{T}_L : DT_L]_{-t}^*) [D\Psi : D\Phi]_t [DT_L : D\tilde{T}_L]_t \\ &= \alpha(h^{it}) \lambda^{\frac{it^2}{2}} \delta^{it} \beta(h^{-it}) = \lambda^{\frac{it^2}{2}} \delta^{it} \alpha(h^{it}) \beta(h^{-it}) \end{aligned}$$

Then, it remains to compute \tilde{P} . If, we put $\tilde{\Phi} = \nu \circ \alpha^{-1} \circ \tilde{T}_L$, we have, for all $t \in \mathbb{R}$ and $x \in \mathcal{N}_{\tilde{T}_L} \cap \mathcal{N}_{\tilde{\Phi}}$:

$$\begin{aligned} \tilde{P}^{it} \Lambda_{\tilde{\Phi}}(x) &= \tilde{\lambda}^{t/2} \Lambda_{\tilde{\Phi}}(\tilde{\tau}_t(x)) = \lambda^{t/2} \Lambda_{\Phi}(\tau_t(x) \beta(h^{1/2})) = \lambda^{t/2} \Lambda_{\Phi}(\tau_t(x) \beta(h^{1/2})) \\ &= P^{it} \Lambda_{\Phi}(x \beta(h^{1/2})) = P^{it} \Lambda_{\tilde{\Phi}}(x) \end{aligned}$$

□

THEOREM 5.43. — *Let $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$ and $(N, M, \alpha, \beta, \Gamma, \nu', T'_L, T'_R)$ be measured quantum groupoids such that there exist strictly positive operators h and k affiliated with N which strongly commute and $[D\nu' : D\nu]_t = k^{\frac{it^2}{2}} h^{it}$ for all $t \in \mathbb{R}$. For all $t \in \mathbb{R}$, fundamental objects of the two structures are linked by:*

- i) $R' = R$
- ii) $\tau'_t = Ad_{\alpha(k^{\frac{-it^2}{2}} h^{-it}) \beta(k^{\frac{it^2}{2}} h^{it})} \circ \tau_t = Ad_{\alpha([D\nu' : D\nu]_t^*) \beta([D\nu' : D\nu]_t)} \circ \tau_t$
- iii) $\lambda' = \lambda$
- iv) $\delta' = \delta$ where δ and δ' have been defined in proposition 5.11
- v) $P'^{it} = \alpha(k^{\frac{it^2}{2}} h^{it}) \beta(k^{\frac{-it^2}{2}} h^{-it}) J_{\Phi} \alpha(k^{\frac{it^2}{2}} h^{it}) \beta(k^{\frac{-it^2}{2}} h^{-it}) J_{\Phi} P^{it}$

Proof. — We successively apply the two previous propositions. □

We state results of the section in the following theorems:

THEOREM 5.44. — *Let $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$ be a measured quantum groupoid. If T' is a left invariant operator-valued weight which is β -adapted w.r.t ν , then there exists a strictly positive operator h affiliated with $Z(N)$ such that, for all $t \in \mathbb{R}$:*

$$\nu \circ \alpha^{-1} \circ T' = (\nu \circ \alpha^{-1} \circ T_L)_{\beta(h)}$$

We have a similar result for the right invariant operator-valued weights.

THEOREM 5.45. — *Let $(N, M, \alpha, \beta, \Gamma, \nu, T_L, R \circ T_L \circ R)$ be a measured quantum groupoid. Then there exists a strictly positive operator δ affiliated with $M \cap \alpha(N)' \cap \beta(N)'$ called modulus and then there exists a strictly positive operator λ affiliated with $Z(M) \cap \alpha(N) \cap \beta(N)$ called scaling operator such that $[D\nu \circ \alpha^{-1} \circ T_L \circ R : D\nu \circ \alpha^{-1} \circ T_L]_t = \lambda^{\frac{it}{2}} \delta^{it}$ for all $t \in \mathbb{R}$.*

Moreover, we have, for all $s, t \in \mathbb{R}$:

- $[D\nu \circ \alpha^{-1} \circ T_L \circ \tau_s : D\nu \circ \alpha^{-1} \circ T_L]_t = \lambda^{-ist}$
- $[D\nu \circ \alpha^{-1} \circ T_L \circ R \circ \tau_s : D\nu \circ \alpha^{-1} \circ T_L \circ R]_t = \lambda^{-ist}$
- i) $[D\nu \circ \alpha^{-1} \circ T_L \circ \sigma_s^{\nu \circ \alpha^{-1} \circ T_L \circ R} : D\nu \circ \alpha^{-1} \circ T_L]_t = \lambda^{ist}$
- $[D\nu \circ \alpha^{-1} \circ T_L \circ R \circ \sigma_s^{\nu \circ \alpha^{-1} \circ T_L} : D\nu \circ \alpha^{-1} \circ T_L \circ R]_t = \lambda^{-ist}$
- ii) $R(\lambda) = \lambda$, $R(\delta) = \delta^{-1}$, $\tau_t(\delta) = \delta$ and $\tau_t(\lambda) = \lambda$;
- iii) δ is a group-like element i.e $\Gamma(\delta) = \delta \underset{N}{\beta \otimes_\alpha} \delta$.

If ν' is a n.s.f weight on N and h, k are strictly positive operators, affiliated with N , strongly commuting and satisfying $[D\nu' : D\nu]_t = k^{\frac{it}{2}} h^{it}$ for all $t \in \mathbb{R}$, then there exists a n.s.f left invariant operator-valued weight \tilde{T}_L which is β -adapted w.r.t ν' . Moreover, if $(N, M, \alpha, \beta, \Gamma, \nu', T'_L, T'_R)$ is an other measured quantum groupoid, then, for all $t \in \mathbb{R}$, fundamental objects are linked by:

- i) $R' = R$
- ii) $\tau'_t = Ad_{\alpha(k^{\frac{-it}{2}} h^{-it}) \beta(k^{\frac{it}{2}} h^{it})} \circ \tau_t = Ad_{\alpha([D\nu' : D\nu]_t^*) \beta([D\nu' : D\nu]_t)} \circ \tau_t$
- iii) $\lambda' = \lambda$
- iv) $\delta' = \delta$ where δ and δ' have been defined in proposition 5.11
- v) $P'^{it} = \alpha(k^{\frac{it}{2}} h^{it}) \beta(k^{\frac{-it}{2}} h^{-it}) J_\Phi \alpha(k^{\frac{it}{2}} h^{it}) \beta(k^{\frac{-it}{2}} h^{-it}) J_\Phi P^{it}$

6. Examples

6.1. Groupoids. —

DEFINITION 6.1. — A **groupoid** G is a small category in which each morphism $\gamma : x \rightarrow y$ is an isomorphism the inverse of which is γ^{-1} . Let $G^{\{0\}}$ the set of objects of G that we identify with $\{\gamma \in G | \gamma \circ \gamma = \gamma\}$. For all $\gamma \in G$, $\gamma : x \rightarrow y$, we denote $x = \gamma^{-1} \gamma = s(\gamma)$ we call source object and

$y = \gamma\gamma^{-1} = r(\gamma)$ we call range object. If $G^{\{2\}}$ is the set of pairs (γ_1, γ_2) of G such that $s(\gamma_1) = r(\gamma_2)$, then composition of morphisms makes sense in $G^{\{2\}}$.

In [Ren80], J. Renault defines the structure of locally compact groupoid G with a Haar system $\{\lambda^u, u \in G^{\{0\}}\}$ and a quasi-invariant measure μ on $G^{\{0\}}$. We refer to [Ren80] for definitions and notations. We put $\nu = \mu \circ \lambda$. We refer to [Co79] and [ADR00] for discussions about transversal measures.

If G is σ -compact, J.M Vallin constructs in [Val96] two co-involutive Hopf bimodules on the same basis $N = L^\infty(G^{\{0\}}, \mu)$, following T. Yamanouchi's works in [Yam93]. The underlying von Neumann algebras are $L^\infty(G, \nu)$ which acts by multiplication on $H = L^2(G, \nu)$ and $\mathcal{L}(G)$ generated by the left regular representation L of G .

We define a (resp. anti-) representation α (resp. β) from N in $L^\infty(G, \nu)$ such that, for all $f \in N$:

$$\alpha(f) = f \circ r \quad \text{and} \quad \beta(f) = f \circ s$$

For all $i, j \in \{\alpha, \beta\}$, we define $G_{i,j}^{\{2\}} \subset G \times G$ and a measure $\nu_{i,j}^2$ such that:

$$H_{\alpha \otimes_\mu \alpha} H \text{ is identified with } L^2(G_{i,j}^{\{2\}}, \nu_{i,j}^2)$$

For example, $G_{\beta,\alpha}^{\{2\}}$ is equal to $G^{\{2\}}$ and $\nu_{\beta,\alpha}^2$ to ν^2 . Then, we construct a unitary W_G from $H_{\alpha \otimes_\mu \alpha}$ onto $H_{\beta \otimes_\mu \beta}$, defined for all $\xi \in L^2(G_{\alpha,\alpha}^{\{2\}}, \nu_{\alpha,\alpha}^2)$ by:

$$W_G \xi(s, t) = \xi(s, st)$$

for ν^2 -almost all (s, t) in $G^{\{2\}}$.

This leads to define co-products Γ_G and $\widehat{\Gamma}_G$ by formulas:

$$\Gamma_G(f) = W_G(1_{\alpha \otimes_\mu \alpha} f)W_G^* \quad \text{and} \quad \widehat{\Gamma}_G(k) = W_G^*(k_{\beta \otimes_\mu \beta} 1)W_G$$

for all $f \in L^\infty(G, \nu)$ and $k \in \mathcal{L}(G)$, this explicitly gives:

$$\Gamma_G(f)(s, t) = f(st)$$

for all $f \in L^\infty(G, \nu)$ and ν^2 -almost all (s, t) in $G^{\{2\}}$,

$$\widehat{\Gamma}_G(L(h))\xi(x, y) = \int_G h(s)\xi(s^{-1}x, s^{-1}y)d\lambda^{r(x)}(s)$$

for all $\xi \in L^2(G_{\alpha,\alpha}^{\{2\}}, \nu_{\alpha,\alpha}^2)$, h a continuous function with compact support on G and $\nu_{\alpha,\alpha}^2$ -almost all (x, y) in $G_{\alpha,\alpha}^{\{2\}}$. Moreover, we define two co-involutions j_G and \widehat{j}_G by:

$$j_G(f)(x) = f(x^{-1})$$

for all $f \in L^\infty(G, \nu)$ and almost all x ,

$$\widehat{j}_G(g) = Jg^*J$$

for all $g \in \mathcal{L}(G)$ and where J is the involution $J\xi = \bar{\xi}$ for all $\xi \in L^2(G)$. Finally, we define two n.s.f left invariant operator-valued weights P_G and \widehat{P}_G :

$$P_G(f)(y) = \int_G f(x) d\lambda^{r(y)}(x) \quad \text{and} \quad \widehat{P}_G(L(f)) = \alpha(f|_{G^{\{0\}}})$$

for all continuous with compact support f on G ν -almost all y in G .

THEOREM 6.2. — *Let G be a σ -compact, locally compact groupoid with a Haar system and a quasi-invariant measure μ on units. Then:*

$$(L^\infty(G^{\{0\}}, \mu), L^\infty(G, \nu), \alpha, \beta, \Gamma_G, \mu, P_G, j_G P_G j_G)$$

is a commutative measured quantum groupoid and:

$$(L^\infty(G^{\{0\}}, \mu), \mathcal{L}(G), \alpha, \alpha, \widehat{\Gamma}_G, \mu, \widehat{P}_G, \widehat{j}_G \widehat{P}_G \widehat{j}_G)$$

is a symmetric measured quantum groupoid. The unitary $V_G = W_G^$ is the fundamental unitary of the commutative structure.*

Proof. — By [Val96] (th. 3.2.7 and 3.3.7), $(L^\infty(G^{\{0\}}, \mu), L^\infty(G, \nu), \alpha, \beta, \Gamma_G)$ and $(L^\infty(G^{\{0\}}, \mu), \mathcal{L}(G), \alpha, \alpha, \widehat{\Gamma}_G)$ are co-involutive Hopf bimodules with left invariant operator-valued weights; to get right invariants operator-valued weights, we consider $j_G P_G j_G$ and $\widehat{j}_G \widehat{P}_G \widehat{j}_G$.

Since $L^\infty(G, \nu)$ is commutative, P_G is adapted w.r.t μ by [Val96] (theorem 3.3.4), $\sigma_t^{\mu \circ \alpha^{-1} \circ \widehat{P}_G}$ fixes point by point $\alpha(N)$ so that \widehat{P}_G is adapted w.r.t μ .

Finally, for all e, f, g continuous functions with compact support and almost all (s, t) in $G^{\{2\}}$, we have, by 3.16:

$$\begin{aligned} (1 \otimes_{\beta \otimes_{\alpha} \mu} J e J) W_G(f \otimes_{\alpha \otimes_{\alpha} \mu} g)(s, t) &= \overline{e(t)} f(s) g(st) = \Gamma_G(g)(f \otimes_{\beta \otimes_{\alpha} \mu} \bar{e})(s, t) \\ &= (1 \otimes_{\beta \otimes_{\alpha} \mu} J e J) U_H(f \otimes_{\alpha \otimes_{\alpha} \mu} g)(s, t) \end{aligned}$$

so that we get $U_H = W_G$. \square

REMARK 6.3. — In the commutative structure, modular function $\frac{d\nu^{-1}}{d\nu}$ and modulus coincide and the scaling operator is trivial.

We have a similar result for measured quantum groupoids in the sense of Hahn ([Hah78a] and [Hah78b]):

THEOREM 6.4. — *From all measured groupoid G , we construct a commutative measured quantum groupoid $(L^\infty(G^{\{0\}}, \mu), L^\infty(G, \nu), \alpha, \beta, \Gamma_G, \mu, P_G, j_G P_G j_G)$ and a symmetric one $(L^\infty(G^{\{0\}}, \mu), \mathcal{L}(G), \alpha, \alpha, \widehat{\Gamma}_G, \mu, \widehat{P}_G, \widehat{j}_G \widehat{P}_G \widehat{j}_G)$. Objects are defined in a similar way as in the locally compact case. The unitary V_G is the fundamental unitary of the commutative structure.*

Proof. — Results come from [Yam93] for the symmetric case. It is sufficient to apply in this case, technics of [Val96] for the commutative case and invariant operator-valued weights. \square

CONJECTURE 6.5. — *If $(N, M, \alpha, \beta, \Gamma, \mu, T_L, T_R)$ is a measured quantum groupoid such that M is commutative, then there exists a locally compact groupoid G such that:*

$$(N, M, \alpha, \beta, \Gamma, \mu, T_L, T_R) \simeq (L^\infty(G^{\{0\}}, \mu), L^\infty(G, \nu), \alpha, \beta, \Gamma_G, \mu, P_G, j_G \circ P_G \circ j_G)$$

6.2. Finite quantum groupoids. —

DEFINITION 6.6. — (Weak Hopf C*-algebras [BSz96]) We call **weak Hopf C*-algebra** or finite quantum groupoid all $(M, \Gamma, \kappa, \varepsilon)$ where M is a finite dimensional C*-algebra with a co-product $\Gamma : M \rightarrow M \otimes M$, a co-unit ε and an antipode $\kappa : M \rightarrow M$ such that, for all $x, y \in M$:

- i) Γ is a *-homomorphism (not necessary unital);
- ii) Unit and co-unit satisfy the following relation:

$$(\varepsilon \otimes \varepsilon)((x \otimes 1)\Gamma(1)(1 \otimes y)) = \varepsilon(xy)$$

- iii) κ is an anti-homomorphism of algebra and co-algebra such that:

$$\begin{aligned} - (\kappa \circ *)^2 &= \iota \\ - (m(\kappa \otimes id) \otimes id)(\Gamma \otimes id)\Gamma(x) &= (1 \otimes x)\Gamma(1). \end{aligned}$$

where m denote the product on M .

We recall some results [NV00], [NV02] and [BNS99]. If $(M, \Gamma, \kappa, \varepsilon)$ is a weak Hopf C*-algebra. We call co-unit range (resp. source) the application $\varepsilon_t = m(id \otimes \kappa)\Gamma$ (resp. $\varepsilon_s = m(\kappa \otimes id)\Gamma$). We have $\kappa \circ \varepsilon_t = \varepsilon_s \circ \kappa$. There exists a unique faithful positive linear form h , called normalized Haar measure of $(M, \Gamma, \kappa, \varepsilon)$ which is κ -invariant, such that $(id \otimes h)(\Gamma(1)) = 1$ and, for all $x, y \in M$, we have:

$$(id \otimes h)((1 \otimes y)\Gamma(x)) = \kappa((i \otimes h)(\Gamma(y)(1 \otimes x)))$$

Moreover, $E_h^s = (h \otimes id)\Gamma$ (resp. $E_h^t = (id \otimes h)\Gamma$) is a Haar conditional expectation to the source (resp. range) Cartan subalgebra $\varepsilon_s(M)$ (resp. range $\varepsilon_t(M)$) such that $h \circ E_h^s = h$ (resp. $h \circ E_h^t = h$). Range and source Cartan subalgebras commute.

By [Val03] and [Nik02], we can always assume that $\kappa_{|\varepsilon_t(M)}^2 = id$ thanks to a deformation. **In the following, we assume that the condition holds.**

Since $h \circ \kappa = h$ and $\kappa \varepsilon_t = \varepsilon_s \kappa$, we have $h \circ \varepsilon_t = h \circ \varepsilon_s$.

THEOREM 6.7. — *Let $(M, \Gamma, \kappa, \varepsilon)$ be a weak Hopf C*-algebra, h its normalized Haar measure, E_h^s (resp. E_h^t) its source (resp. range) Haar conditional expectation and $\varepsilon_t(M)$ its range Cartan subalgebra. We put $N = \varepsilon_t(M)$, $\alpha = id|_N$,*

$\beta = \kappa|_N$, $\tilde{\Gamma}$ the co-product Γ viewed as an operator which takes value in:

$$M \underset{N}{\beta \star \alpha} M \simeq (M \otimes M)_{\Gamma(1)}$$

and $\mu = h \circ \alpha = h \circ \beta$. Then $(N, M, \alpha, \beta, \tilde{\Gamma}, \mu, E_h^t, E_h^s)$ is a measured quantum groupoid.

Proof. — α is a representation from N in M and, since $\kappa_{|\varepsilon_t(M)}^2 = id$, β is a anti-representation from N in M . They commute each other because Cartan subalgebras commute and $\kappa \circ \varepsilon_t = \varepsilon_s \circ \kappa$. For all $n \in N$, there exists $m \in M$ such that $n = \varepsilon_t(m)$. So, we have:

$$\tilde{\Gamma}(\alpha(n)) = \tilde{\Gamma}(\varepsilon_t(m)) = \Gamma(1)(\varepsilon_t(m) \otimes 1)\Gamma(1) = \alpha(n) \underset{N}{\beta \otimes \alpha} 1$$

Also, we have $\tilde{\Gamma}(\beta(n)) = 1 \underset{N}{\beta \otimes \alpha} \beta(n)$ and $\tilde{\Gamma}$ is a co-product. Then $(N, M, \alpha, \beta, \Gamma)$ is Hopf bimodule. Moreover, for all $n \in N$ and $t \in \mathbb{R}$, we have:

$$\begin{aligned} \sigma_t^{E_h^t}(\beta(n)) &= \sigma_t^{h \circ E_h^t}(\beta(n)) = \sigma_t^{h \circ E_h^s}(\beta(n)) = \sigma_t^{h|_{\beta(N)}}(\beta(n)) \\ &= \beta(\sigma_{-t}^{h|_{\beta(N)} \circ \beta}(n)) = \beta(\sigma_{-t}^\mu(n)) \end{aligned}$$

and E_h^t is β -adapted w.r.t μ . Since $E_h^s = \kappa \circ E_h^t \circ \kappa$, then E_h^s is α -adapted w.r.t μ . \square

THEOREM 6.8. — *Let $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$ be a measured quantum groupoid such that M is finite dimensional. Then, there exist $\tilde{\Gamma}$, κ and ε such that $(M, \tilde{\Gamma}, \kappa, \varepsilon)$ is a weak Hopf C^* -algebra.*

Proof. — By 2.3, we identify via $I_{\beta, \alpha}^\nu$, $L^2(M) \underset{N}{\beta \otimes \alpha} L^2(M)$ with a subspace of $L^2(M) \otimes L^2(M)$. We put $\tilde{\Gamma}(x) = I_{\beta, \alpha}^\nu \Gamma(x) (I_{\beta, \alpha}^\nu)^*$. By [Val01] (definition 2.2.3), the fundamental pseudo-multiplicative unitary becomes a multiplicative partial isometry on $L^2(M) \otimes L^2(M)$ of basis $(N, \alpha, \hat{\beta}, \beta)$ by $I = I_{\alpha, \hat{\beta}}^\nu W(I_{\beta, \alpha}^\nu)^*$. I is regular in the sense of [Val01] (definition 2.6.3) by 5.33. Moreover, if we put $H = L^2(M)$, then $Tr_H(R(m)) = Tr_H(m)$ for all $m \in M$ because R is implemented by an anti-unitary, so $Tr_H \circ \beta = Tr_H \circ \alpha = Tr_H \circ \hat{\beta}$ and we conclude by [Val01] (proposition 3.1.3). \square

REMARK 6.9. — With notations of section 2.3, κ and S are linked by:

$$\kappa(x) = \alpha(n_o^{1/2} d^{1/2}) \beta(n_o^{-1/2} d^{-1/2}) S(x) \alpha(n_o^{-1/2} d^{-1/2}) \beta(n_o^{1/2} d^{1/2})$$

6.3. Quantum groups. —

THEOREM 6.10. — *Measured quantum groupoids, basis N on which is equal to \mathbb{C} are exactly locally compact quantum groups (in the von Neumann setting) introduced by J. Kustermans and S. Vaes in [KV03].*

Proof. — In this case, the notion of relative tensor product is just usual tensor product of Hilbert spaces, the notion of fibered product is just tensor product of von Neumann algebras and the notion of operator-valued weight is just weight. \square

6.4. **Compact case.** — In this section, we show that pseudo-multiplicative unitaries of compact type in the sense of [Eno02] correspond exactly to measured quantum groupoids with a Haar conditional expectation.

DEFINITION 6.11. — Let W be a pseudo-multiplicative unitary over N w.r.t $\alpha, \beta, \hat{\beta}$. Let ν be a n.s.f weight on N . We say that W is of **compact type** w.r.t ν if there exists $\xi \in H$ such that:

- i) ξ belongs to $D(H_{\hat{\beta}}, \nu^o) \cap D({}_\alpha H, \nu) \cap D(H_\beta, \nu^o)$;
- ii) $\langle \xi, \xi \rangle_{\hat{\beta}, \nu^o} = \langle \xi, \xi \rangle_{\alpha, \nu} = \langle \xi, \xi \rangle_{\beta, \nu^o} = 1$
- iii) we have $W(\xi \otimes_{\hat{\beta}} \eta) = \xi \otimes_{\alpha} \eta$ for all $\eta \in H$.

In this case, ξ is said to be **fixed and bi-normalized**. We also say that W is of **discrete type** w.r.t ν if \hat{W} is of compact type.

By [Eno02] (proposition 5.11), we recall that, if W is of compact type w.r.t ν and ξ is a fixed and bi-normalized vector, then ν shall be a faithful, normal, positive form on N which is equal to $\omega_\xi \circ \alpha = \omega_\xi \circ \beta = \omega_\xi \circ \hat{\beta}$ and it is called **canonical form**.

PROPOSITION 6.12. — *Let $(N, M, \alpha, \beta, \Gamma)$ be a Hopf bimodule. Assume there exist:*

- i) *a n.f left invariant conditional expectation from E to $\alpha(N)$;*
- ii) *a n.f right invariant conditional expectation from F to $\beta(N)$;*
- iii) *a n.f state ν on N such that $\nu \circ \alpha^{-1} \circ E = \nu \circ \beta^{-1} \circ F$.*

Then $(N, M, \alpha, \beta, \Gamma, \nu, E, F)$ is a measured quantum groupoid. Moreover, if R, τ, λ and δ are fundamental objects of the structure, then we have $F = R \circ E \circ R$ and $\lambda = \delta = 1$. Finally, $\Lambda_{\nu \circ \alpha^{-1} \circ E}(1)$ is co-fixed and bi-normalized, and the fundamental pseudo-multiplicative unitary W is weakly regular and of discrete type in sense of [Eno02] (paragraphe 5).

Proof. — For all $t \in \mathbb{R}$ and $n \in N$, we have:

$$\sigma_t^E(\beta(n)) = \sigma_t^{\nu \circ \alpha^{-1} \circ E}(\beta(n)) = \sigma_t^{\nu \circ \beta^{-1} \circ F}(\beta(n)) = \beta(\sigma_{-t}^\nu(n))$$

Also, we have:

$$\sigma_t^F(\alpha(n)) = \sigma_t^{\nu \circ \beta^{-1} \circ F}(\alpha(n)) = \sigma_t^{\nu \circ \alpha^{-1} \circ E}(\alpha(n)) = \alpha(\sigma_t^\nu(n))$$

so that $(N, M, \alpha, \beta, \Gamma, \nu, E, F)$ is a measured quantum groupoid. By definition, we have:

$$[D\nu \circ \alpha^{-1} \circ E \circ R : D\nu \circ \alpha^{-1} \circ E]_t = \lambda^{\frac{it^2}{2}} \delta^{it}$$

On the other hand, since $\nu \circ \alpha^{-1} \circ E = \nu \circ \beta^{-1} \circ F$ and by uniqueness, there exists a strictly positive element h affiliated with $Z(N)$:

$$[D\nu \circ \alpha^{-1} \circ E \circ R : D\nu \circ \alpha^{-1} \circ E]_t = [DR \circ E \circ R : DF]_t = \alpha(h^{it})$$

We deduce that $\lambda = 1$ and $\delta = \alpha(h)$, so $\alpha(h^{-1}) = \delta^{-1} = R(\delta) = \beta(h)$ and by [Eno00] (5.2), we get $h = 1$.

We put $\Phi = \nu \circ \alpha^{-1} \circ E$. If $(\xi_i)_{i \in I}$ is a (N°, ν°) -basis of $(H_\Phi)_\beta$ then, for all $v \in D(H_\beta, \nu^\circ)$:

$$\begin{aligned} U_H(v \underset{\nu^\circ}{\alpha \otimes \beta} \Lambda_\Phi(1)) &= \sum_{i \in I} \xi_i \underset{\nu}{\beta \otimes \alpha} \Lambda_\Phi((\omega_{v, \xi_i} \underset{\nu}{\beta \star \alpha} id)(\Gamma(1))) \\ &= \sum_{i \in I} \xi_i \underset{\nu}{\beta \otimes \alpha} \alpha(< v, \xi_i >_{\beta, \nu^\circ}) \Lambda_\Phi(1) = v \underset{\nu}{\beta \otimes \alpha} \Lambda_\Phi(1) \end{aligned}$$

It is easy to see that $\Lambda_\Phi(1)$ belongs to $D((H_\Phi)_\beta, \nu^\circ) \cap D(\alpha H_\Phi, \nu)$ and satisfies $< \Lambda_\Phi(1), \Lambda_\Phi(1) >_{\beta, \nu^\circ} = < \Lambda_\Phi(1), \Lambda_\Phi(1) >_{\alpha, \nu} = 1$ so that, by continuity, we get $U_H(v \underset{\nu^\circ}{\alpha \otimes \beta} \Lambda_\Phi(1)) = v \underset{\nu}{\beta \otimes \alpha} \Lambda_\Phi(1)$ for all $v \in H$ i.e $\Lambda_\Phi(1)$ is co-fixed and bi-normalized. Since $\nu \circ \alpha^{-1} \circ E = \Phi = \nu \circ \beta^{-1} \circ F$, we have by 3.6, for all $n \in \mathcal{N}_\nu$:

$$\beta(n^*) \Lambda_\Phi(1) = \beta(n^*) J_\Phi \Lambda_\Phi(1) = J_\Phi \Lambda_F(1) \Lambda_\nu(n)$$

so that $\Lambda_\Phi(1)$ is β -bounded w.r.t ν° and $R^{\beta, \nu^\circ}(\Lambda_\Phi(1)) = J_\Phi \Lambda_F(1) J_\nu$. Consequently, $\Lambda_\Phi(1)$ is bi-normalized and W is of discrete type. \square

COROLLARY 6.13. — *Let W be a weakly regular pseudo-multiplicative unitary over N w.r.t $\alpha, \beta, \hat{\beta}$ of compact type w.r.t the canonical form ν . If ξ a fixed and bi-normalized vector, we put:*

- i) \mathcal{A} the von Neumann algebra generated by right leg of W ;
- ii) $\Gamma(x) = \sigma_{\nu^\circ} W(x \underset{\nu^\circ}{\alpha \otimes \beta} 1) W^* \sigma_\nu$ for all $x \in \mathcal{A}$;
- iii) $E = (\omega_\xi \underset{\nu}{\beta \star \alpha} id) \circ \Gamma$ and $F = (id \underset{\nu}{\beta \star \alpha} \omega_\xi) \circ \Gamma$.

Then $(N, \mathcal{A}, \alpha, \beta, \Gamma, \nu, E, F)$ is a measured quantum groupoid. Moreover, if R, τ, λ and δ are the fundamental objects of the structure, we have $F = R \circ E \circ R$, $\lambda = \delta = 1$ and the fundamental unitary is \tilde{W} .

Proof. — By [EV00] (6.3), we know that $(N, \mathcal{A}, \alpha, \beta, \Gamma)$ is a Hopf bimodule. By [Eno02] (theorem 6.6), E is a n.f left invariant conditional expectation from \mathcal{A} to $\alpha(N)$. By [Eno02] (propositions 6.2 and 6.3), F is a n.f right invariant conditional expectation from \mathcal{A} to $\beta(N)$. Moreover, we clearly have $\omega_\xi \circ E = \omega_\xi \circ F$ so that $\nu \circ \alpha^{-1} \circ E = \nu \circ \beta^{-1} \circ F$. We are in conditions of the previous proposition and we get that $(N, \mathcal{A}, \alpha, \beta, \Gamma, \nu, E, F)$ is a measured quantum groupoid, $F = R \circ E \circ R$ and $\lambda = \delta = 1$. Finally, by [Eno02] (corollaire 7.7), \hat{W} is the fundamental unitary. (More exactly, it is $\sigma_{\nu^*} W_s^* \sigma_\nu$ where W_s is the standard form of W in the sense of [Eno02] (paragraphe 7)). \square

The converse is also true and so we characterize the compact case:

COROLLARY 6.14. — *Let $(N, M, \alpha, \beta, \Gamma)$ be a Hopf bimodule. We assume there exist:*

- i) a co-involution R ;
- ii) a n.f left invariant conditional expectation from E to $\alpha(N)$.

Then there exists a n.f state ν on N such that $(N, M, \alpha, \beta, \Gamma, \nu, E, R \circ E \circ R)$ is a measured quantum groupoid with trivial modulus and scaling operator and the fundamental unitary of which is of discrete type w.r.t ν .

Proof. — We put $F = R \circ E \circ R$ which is a n.f right invariant conditional expectation from M to $\beta(N)$. We also put:

$$\tilde{E} = E|_{\beta(N)} : \beta(N) \rightarrow \alpha(Z(N)) \text{ and } \tilde{F} = F|_{\alpha(N)} : \alpha(N) \rightarrow \beta(Z(N))$$

We have, for all $m \in M$:

$$\begin{aligned} \tilde{F}E(m) \underset{N}{\beta \otimes_\alpha} 1 &= (F \underset{N}{\beta \star_\alpha} id)(E(m) \underset{N}{\beta \otimes_\alpha} 1) \\ &= (F \underset{N}{\beta \star_\alpha} id)(id \underset{N}{\beta \star_\alpha} E)\Gamma(m) \\ &= (id \underset{N}{\beta \star_\alpha} E)(F \underset{N}{\beta \star_\alpha} id)\Gamma(m) \\ &= (id \underset{N}{\beta \star_\alpha} E)(1 \underset{N}{\beta \otimes_\alpha} F(m)) = 1 \underset{N}{\beta \otimes_\alpha} \tilde{E}F(m) \end{aligned}$$

so, if $\tilde{F}E(m) = \beta(n)$ for some $n \in Z(N)$, then $\tilde{E}F(m) = \alpha(n)$. Moreover, we have:

$$\begin{aligned} \tilde{E}F(m) \underset{N}{\beta \otimes_\alpha} 1 &= EF(m) \underset{N}{\beta \otimes_\alpha} 1 = (id \underset{N}{\beta \star_\alpha} E)\Gamma(F(m)) \\ &= (id \underset{N}{\beta \star_\alpha} E)(1 \underset{N}{\beta \otimes_\alpha} F(m)) = 1 \underset{N}{\beta \otimes_\alpha} \tilde{E}F(m) \end{aligned}$$

so that $\alpha(n) = \beta(n)$. Consequently $\tilde{E}F(m) = \tilde{F}E(m)$ and $EF = FE$ is a n.f conditional expectation from M to:

$$\tilde{N} = \alpha(\{n \in Z(N), \alpha(n) = \beta(n)\}) = \beta(\{n \in Z(N), \alpha(n) = \beta(n)\})$$

Also, we have $R|_{\tilde{N}} = id$. So, if ω is a n.f state on \tilde{N} , we have $\omega \circ \tilde{E} \circ \beta = \omega \circ \tilde{F} \circ \alpha$ and $\nu = \omega \circ \tilde{E} \circ \beta = \omega \circ \tilde{F} \circ \alpha$ satisfies hypothesis of 6.12: then, corollary holds. \square

COROLLARY 6.15. — *Let $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$ be a measured quantum groupoid such that T_L is a conditional expectation. Then there exists a n.f state ν' on N such that $\sigma^{\nu'} = \sigma^\nu$ and the fundamental unitary is of discrete type w.r.t ν' .*

Proof. — Let R be the co-involution. By the previous corollary, there exists a n.f state ν' on N such that $(N, M, \alpha, \beta, \Gamma, \nu', T_L, R \circ T_L \circ R)$ is a measured quantum groupoid. Since T_L is β -adapted w.r.t ν and ν' , we have $\sigma^{\nu'} = \sigma^\nu$. We easily verify that the fundamental unitary of the first structure coincides with that of the last one which is of discrete type w.r.t ν' by the previous corollary. \square

6.5. Depth 2 inclusions. — Let $M_0 \subseteq M_1$ be an inclusion of von Neumann algebras. We call **basis construction** the following inclusions:

$$M_0 \subseteq M_1 \subseteq M_2 = J_1 M'_0 J_1 = \text{End}_{M'_0}(L^2(M_1))$$

By iteration, we construct Jones' tower $M_0 \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq \dots$

DEFINITION 6.16. — If $M'_0 \cap M_1 \subseteq M'_0 \cap M_2 \subseteq M'_0 \cap M_3$ is a basis construction, then the inclusion is said to be **of depth 2**.

Let T_1 be a n.s.f operator-valued weight from M_1 to M_0 . By Haagerup's construction [Str81] (12.11) and [EN96] (10.1), it is possible to define a canonical n.s.f operator-valued weight T_2 from M_2 to M_1 such that, for all $x, y \in \mathcal{N}_{T_1}$, we have:

$$T_2(\Lambda_{T_1}(x)\Lambda_{T_1}(y)^*) = xy^*$$

By iteration, we define, for all $i \geq 1$, a n.s.f operator-valued weight T_i from M_i to M_{i-1} . If ψ_0 is n.s.f weight sur M_0 , we put $\psi_i = \psi_{i-1} \circ T_i$.

DEFINITION 6.17. — [EN96] (11.12), [EV00] (3.6). T_1 is said to be regular if restrictions of T_2 to $M'_0 \cap M_2$ and of T_3 to $M'_1 \cap M_3$ are semifinite.

PROPOSITION 6.18. — [EV00] (3.2, 3.8, 3.10). *If $M_0 \subset M_1$ is an inclusion with a regular n.s.f operator-valued weight T_1 from M_1 to M_0 , then there exists a natural *-representation π of $M'_0 \cap M_3$ on $L^2(M'_0 \cap M_2)$ whose restriction to $M'_0 \cap M_2$ is the standard representation of $M'_0 \cap M_2$. Moreover, the inclusion is of depth 2 if, and only if π is faithful.*

The following theorem exhibits a structure of measured quantum groupoid coming from inclusion of von Neumann algebras.

THEOREM 6.19. — [Eno04] (theorem 9.2) *Let $M_0 \subset M_1$ be a depth 2 inclusion of von Neumann algebras with a regular n.s.f operator-valued weight T_1 from M_1 to M_0 . Let us assume there exists a n.s.f weight χ on $M'_0 \cap M_1$ such that $\sigma_t^\chi = \sigma_t^{T_1}$ for all $t \in \mathbb{R}$.*

Then, there exists Γ such that $(M'_1 \cap M_2, M'_1 \cap M_3, j_2, id, \Gamma)$ is a Hopf bimodule with a left invariant operator-valued weight $T_{3|M'_1 \cap M_3}$ which is j_2 -adapted and a co-involution j_2 where j_2 is the canonical anti-isomorphism from M_2 onto M'_2 which sends x to $J_2 x^ J_2$ where J_2 is given by Tomita's theory on $L^2(M_2)$. So, we get a structure of measured quantum groupoid on $M'_1 \cap M_3$.*

By [EV00], theorem 7.3 and proposition 7.5, the previous quantum groupoid acts on the von Neumann algebra M_1 such that invariants are exactly elements of the von Neumann algebra M_0 .

REMARK 6.20. — If M_0 and M_1 are semi-finite, then σ^{T_1} is interior. Moreover, if, $M'_0 \cap M_1$ is semi-finite, then σ^{T_1} becomes the modular group of a n.s.f weight χ on $M'_0 \cap M_1$. Consequently, all operator-valued weights are adapted in sense of [Eno00]. Then, we can also put a structure of measured quantum groupoid on $M'_0 \cap M_2$ such that the left operator-valued weight $T_{2|M'_0 \cap M_2}$ is invariant and j_1 -adapted where j_1 comes from Tomita's theory. This situation will be developed in more details in a forthcoming article about duality.

6.6. Quantum space quantum groupoid. — Let M be a von Neumann algebra. M acts on $H = L^2(M) = L^2_\nu(M)$ where ν is a n.s.f weight on M . We denote by M' , (resp. $Z(M)'$) the commutant of M (resp. $Z(M)$) in $\mathcal{L}(L^2(M))$. Let tr be a n.s.f trace on $Z(M)$. $M' \star_{Z(M)} M = M' \otimes_{Z(M)} M$ acts on $L^2(M) \otimes_{tr} L^2(M)$. There exists a n.s.f operator-valued weight T from M to $Z(M)$ such that $\nu = tr \circ T$.

Let α (resp. β) be the (resp. anti-) representation of M to $M' \otimes_{Z(M)} M$ such that $\alpha(m) = 1 \otimes_{Z(M)} m$ (resp. $\beta(m) = j(m) \otimes_{Z(M)} 1$) where $j(x) = J_\nu x^* J_\nu$ for all $x \in \mathcal{L}(L^2_\nu(M))$.

PROPOSITION 6.21. — *The following formula:*

$$I : [L^2(M) \otimes_{tr} L^2(M)] \otimes_{\nu} [L^2(M) \otimes_{tr} L^2(M)] \rightarrow L^2(M) \otimes_{tr} L^2(M) \otimes_{tr} L^2(M)$$

$$[\Lambda_\nu(y) \otimes_{tr} \eta] \otimes_{\nu} \Xi \mapsto \alpha(y) \Xi \otimes_{tr} \eta$$

for all $\eta \in L^2(M), \Xi \in L^2(M) \otimes_{tr} L^2(M)$ and $y \in M$, defines a canonical isomorphism such that we have $I([m \otimes_{tr} z] \otimes_{\nu} Z) = (\alpha(M) Z \otimes_{Z(M)} z) I$, for all $m \in M, z \in Z(M)'$ and $Z \in \mathcal{L}(L^2(M)) \star_{Z(M)} M'$.

Proof. — Straightforward. \square

We identify $(M' \otimes_{Z(M)} M)_{\beta \star_{\alpha} M}$ with $M' \otimes_{Z(M)} Z(M) \otimes_{Z(M)} M$ and so with $M' \otimes_{Z(M)} M$. We define a normal $*$ -homomorphism Γ by:

$$\begin{aligned} M' \otimes_{Z(M)} M &\rightarrow (M' \otimes_{Z(M)} M)_{\beta \star_{\alpha} M} \\ n \otimes_{Z(M)} m &\mapsto I^*(n \otimes_{Z(M)} 1 \otimes_{Z(M)} m)I = [1 \otimes_{Z(M)} m]_{\beta \otimes_{\alpha} \nu} [n \otimes_{Z(M)} 1] \end{aligned}$$

Γ is, in fact, the identity through the previous isomorphism.

THEOREM 6.22. — *If we put $T_R = id_{Z(M)} \star T$ and $R = \varsigma_{Z(M)} \circ (j \otimes_{Z(M)} j)$, then $(M, M' \otimes_{Z(M)} M, \alpha, \beta, \Gamma, \nu, R \circ T_R \circ R, T_R)$ becomes a measured quantum groupoid w.r.t ν called **quantum space quantum groupoid**.*

Proof. — By definition, Γ is a morphism of Hopf bimodule. We have to prove co-product relation. For all $m \in M$ and $n \in M'$, we have:

$$\begin{aligned} (\Gamma_{\beta \star_{\alpha} \nu} id) \circ \Gamma(n \otimes_{Z(M)} m) &= [1 \otimes_{Z(M)} m]_{\beta \otimes_{\alpha} \nu} [1 \otimes_{Z(M)} 1]_{\beta \otimes_{\alpha} \nu} [n \otimes_{Z(M)} 1] \\ &= (id_{\beta \star_{\alpha} \nu} \Gamma) \circ \Gamma(n \otimes_{Z(M)} m) \end{aligned}$$

Now, we show that T_R is right invariant and α -adapted w.r.t ν . So, for all $m \in M, n \in M'$ and $\xi \in D(\alpha(L^2(M) \otimes_{tr} L^2(M)), \nu^o)$, we put $\Psi = \nu \circ \beta^{-1} \circ T_R$ and we compute:

$$\begin{aligned} \omega_{\xi}((\Psi_{\beta \star_{\alpha} \nu} id) \Gamma(n \otimes_{Z(M)} m)) &= \Psi((id_{\beta \star_{\alpha} \nu} \omega_{\xi})([1 \otimes_{Z(M)} m]_{\beta \otimes_{\alpha} \nu} [n \otimes_{Z(M)} 1])) \\ &= \Psi([1 \otimes_{Z(M)} m] \beta(< [n \otimes_{Z(M)} 1] \xi, \xi >_{\alpha, \nu})) \\ &= \nu(< [n \otimes_{Z(M)} T(m)] \xi, \xi >_{\alpha, \nu}) \\ &= \omega_{\xi}(n \otimes_{Z(M)} T(m)) = \omega_{\xi}(T_R(n \otimes_{Z(M)} m)) \end{aligned}$$

Finally, we have for all $t \in \mathbb{R}$:

$$\begin{aligned} \sigma_t^{T_R} &= \sigma_t^{\nu' \star_{Z(M)} \nu} |_{(M' \otimes_{Z(M)} M) \cap \beta(M)'} = \sigma_t^{\nu' \star_{Z(M)} \nu} |_{(M' \otimes_{Z(M)} M) \cap (M \star_{Z(M)} \mathcal{L}(L^2(M)))} \\ &= \sigma_t^{\nu' \star_{Z(M)} \nu} |_{Z(M) \otimes_{Z(M)} M} = (id \otimes_{Z(M)} \sigma_t^{\nu}) |_{1 \otimes_{Z(M)} M} = 1 \otimes_{Z(M)} \sigma_t^{\nu} \end{aligned}$$

so that $\sigma_t^{T_R} \circ \alpha(m) = 1 \otimes_{Z(M)} \sigma_t^{\nu}(m) = \alpha(\sigma_t^{\nu}(m))$ for all $t \in \mathbb{R}$ and $m \in M$.

Since it is easy to see that R is a co-involution, we have done. \square

By 3.16, we can compute the pseudo-multiplicative unitary. Let first notice that $\Phi = \nu' \star_{Z(M)} \nu = \Psi$ so that $\lambda = \delta = 1$ and:

$$\alpha = 1 \otimes_{Z(M)} id, \hat{\alpha} = id \otimes_{Z(M)} 1, \beta = j \otimes_{Z(M)} 1 \text{ and } \hat{\beta} = 1 \otimes_{Z(M)} j$$

For example, we have $D((H \otimes_{tr} H)_{\hat{\beta}, \nu^o}) \supset H \otimes_{tr} D(H_j, \nu^o) = H \otimes_{tr} \Lambda_\nu(\mathcal{N}_\nu)$ and for all $\eta \in H$ and $y \in \mathcal{N}_\nu$, we have $R^{\hat{\beta}, \nu^o}(\eta \otimes_{tr} \Lambda_\nu(y)) = \lambda_\eta^{tr} R^{j, \nu^o}(\Lambda_\nu(y)) = \lambda_\eta^{tr} y$.

LEMMA 6.23. — *We have, for all $\eta \in H$ and $e \in \mathcal{N}_\nu$:*

$$I\rho_{\eta \otimes_{tr} J_\nu \Lambda_\nu(e)}^{\beta, \alpha} = \lambda_\eta^{tr} J_\nu e J_\nu \otimes_{Z(M)} 1 \text{ and } I\lambda_{\Lambda_\nu(y) \otimes \eta}^{\beta, \alpha} = \rho_\eta^{tr} (1 \otimes_{Z(M)} y)$$

Proof. — Straightforward. \square

PROPOSITION 6.24. — *We have, for all $\Xi \in H \otimes_{tr} H, \eta \in H$ and $m \in \mathcal{N}_\nu$:*

$$W^*(\Xi \otimes_{\nu^o} \hat{\beta} (\eta \otimes_{tr} \Lambda_\nu(m))) = I^*(\eta \otimes_{tr} (1 \otimes_{Z(M)} m) \Xi)$$

Proof. — For all $m, e \in \mathcal{N}_\nu$ and $m', e' \in \mathcal{N}_{\nu'}$, we have by the previous lemma:

$$\begin{aligned} I\Gamma(m' \otimes_{Z(M)} m) \rho_{J_{\nu'} \Lambda_{\nu'}(e') \otimes_{tr} J_\nu \Lambda_\nu(e)}^{\beta, \alpha} &= (m' \otimes_{Z(M)} 1 \otimes_{Z(M)} m) I\rho_{J_{\nu'} \Lambda_{\nu'}(e') \otimes_{tr} J_\nu \Lambda_\nu(e)}^{\beta, \alpha} \\ &= (m' \otimes 1 \otimes m) \lambda_{J_{\nu'} \Lambda_{\nu'}(e')}^{tr} J_\nu e J_\nu \otimes_{Z(M)} 1 \\ &= \lambda_{J_{\nu'} e' J_{\nu'} \Lambda_{\nu'}(m')}^{tr} J_\nu e J_\nu \otimes_{Z(M)} m \end{aligned}$$

On the other hand, we have by 6.21:

$$\begin{aligned} I([1 \otimes_{Z(M)} 1] \otimes_{\nu} \beta \otimes_{\alpha} [J_{\nu'} e' J_{\nu'} \otimes_{Z(M)} J_\nu e J_\nu]) W^* \rho_{\Lambda_{\nu'}(m') \otimes_{tr} \Lambda_{\nu'}(m')}^{\alpha, \hat{\beta}} \\ = (J_{\nu'} e' J_{\nu'} \otimes_{Z(M)} J_\nu e J_\nu \otimes_{Z(M)} 1) I W^* \rho_{\Lambda_{\nu'}(m') \otimes_{tr} \Lambda_{\nu'}(m')}^{\alpha, \hat{\beta}} \end{aligned}$$

Then, by 3.16 and taking the limit over e and e' which go to 1, we get for all $\Xi \in H \otimes_{tr} H$:

$$W^*(\Xi \otimes_{\nu^o} \hat{\beta} (\Lambda_{\nu'}(m') \otimes_{tr} \Lambda_\nu(m))) = I^*(\Lambda_{\nu'}(m') \otimes_{tr} (1 \otimes_{Z(M)} m) \Xi)$$

Now, if $\Xi \in D(\alpha(H \otimes_{tr} H), \nu)$, by continuity and density of $\Lambda_{\nu'}(\mathcal{N}_{\nu'})$ we have for all $\Xi \in D(\alpha(H \otimes_{tr} H), \nu)$:

$$W^*(\Xi \otimes_{\nu^o} \hat{\beta} (\eta \otimes_{tr} \Lambda_\nu(m))) = I^*(\eta \otimes_{tr} (1 \otimes_{Z(M)} m) \Xi)$$

Since $\eta \otimes_{tr} \Lambda_\nu(m) \in D((H \otimes_{tr} H)_{\hat{\beta}, \nu^o})$, the relation holds by continuity for all $\Xi \in H \otimes_{tr} H$. \square

REMARK 6.25. — If σ_{tr} is the flip of $L^2(M) \otimes_{tr} L^2(M)$, then $\sigma_{tr} \circ \hat{\beta} = \beta \circ \sigma_{tr}$ and if $I' = (1 \otimes_{Z(M)} \sigma_{tr}) I(\sigma_{tr} \otimes_{\nu} [1 \otimes_{Z(M)} 1]) \sigma_{\nu^o}$, then I' is the identification:

$$I' : [L^2(M) \otimes_{tr} L^2(M)] \otimes_{\nu^o} [L^2(M) \otimes_{tr} L^2(M)] \rightarrow L^2(M) \otimes_{tr} L^2(M) \otimes_{tr} L^2(M) \\ \Xi \otimes_{\nu} \alpha [\eta \otimes_{tr} \Lambda_{\nu}(y)] \mapsto \eta \otimes_{tr} \alpha(y) \Xi$$

for all $\eta \in L^2(M)$, $\Xi \in L^2(M) \otimes_{tr} L^2(M)$ and $y \in M$. Consequently, by the previous proposition $W^* = I^* I'$.

COROLLARY 6.26. — *We can reconstruct the von Neumann algebra thanks to W :*

$$M' \otimes_{Z(M)} M = \langle (id * \omega_{\xi, \eta})(W^*) \mid \xi \in D((H \otimes_{tr} H)_{\hat{\beta}}, \nu^o), \eta \in D(\alpha(H \otimes_{tr} H), \nu) \rangle^{-w}$$

Proof. — By 3.23, we know that:

$$\langle (id * \omega_{\xi, \eta})(W^*) \mid \xi \in D((H \otimes_{tr} H)_{\hat{\beta}}, \nu^o), \eta \in D(\alpha(H \otimes_{tr} H), \nu) \rangle^{-w} \subset M' \otimes_{Z(M)} M$$

Let $\eta, \xi \in H$ and $m, e \in \mathcal{N}_{\nu}$. Then, for all $\Xi_1, \Xi_2 \in H \otimes_{tr} H$, we have by 6.23:

$$\begin{aligned} & ((id * \omega_{\eta \otimes_{tr} \Lambda_{\nu}(m), \xi \otimes_{tr} J_{\nu} \Lambda_{\nu}(e)})(W^*) \Xi_1 | \Xi_2) \\ &= (W^* (\Xi_1 \otimes_{\nu^o} \alpha \otimes_{\hat{\beta}} [\eta \otimes_{tr} \Lambda_{\nu}(m)]) | \Xi_2 \otimes_{\nu} \alpha [\xi \otimes_{tr} J_{\nu} \Lambda_{\nu}(e)]) \\ &= (I^* (\eta \otimes_{tr} (1 \otimes_{Z(M)} m) \Xi_1) | \Xi_2 \otimes_{\nu} \alpha [\xi \otimes_{tr} J_{\nu} \Lambda_{\nu}(e)]) \\ &= (\eta \otimes_{tr} (1 \otimes_{Z(M)} m) \Xi_1 | \xi \otimes_{tr} (J_{\nu} e J_{\nu} \otimes_{Z(M)} 1) \Xi_2) \\ &= ((\langle \eta, \xi \rangle_{tr} J_{\nu} e^* J_{\nu} \otimes m) \Xi_1 | \Xi_2) \end{aligned}$$

Consequently, we get the reverse inclusion thanks to the relation:

$$(id * \omega_{\eta \otimes_{tr} \Lambda_{\nu}(m), \xi \otimes_{tr} J_{\nu} \Lambda_{\nu}(e)})(W^*) = \langle \eta, \xi \rangle_{tr} J_{\nu} e^* J_{\nu} \otimes_{Z(M)} m$$

□

Now, we compute G so as to get the antipode.

PROPOSITION 6.27. — *If $F_{\nu} = S_{\nu}^*$ comes from Tomita's theory, then we have:*

$$G = \sigma_{tr} \circ (F_{\nu} \otimes_{tr} F_{\nu})$$

Proof. — Let $a = J_\nu a_1 J_\nu \otimes_{Z(M)} a_2, b = J_\nu b_1 J_\nu \otimes_{Z(M)} b_2, c = J_\nu c_1 J_\nu \otimes_{Z(M)} c_2$ and $d = J_\nu d_1 J_\nu \otimes_{Z(M)} d_2$ be elements of $M' \otimes_{Z(M)} M$ analytic w.r.t $\nu' \star_{Z(M)} \nu$. Then, by 6.23, the value of $(\lambda_{\Lambda_\nu(\sigma_{i/2}^\nu(b_1)) \otimes_{tr} \Lambda_\nu(\sigma_{-i}^\nu(b_2^*))}^{\beta, \alpha})^* W^*$ on

$$[\Lambda_{\nu'}(J_\nu a_1 J_\nu) \otimes_{tr} \Lambda_\nu(a_2)] \underset{\nu^o}{\alpha \otimes \hat{\beta}} [\Lambda_{\nu'}(J_\nu d_1^* c_1^* J_\nu) \otimes_{tr} \Lambda_\nu(d_2^* c_2^*)]$$

is equal to:

$$\begin{aligned} & (\lambda_{\Lambda_\nu(\sigma_{i/2}^\nu(b_1)) \otimes_{tr} \Lambda_\nu(\sigma_{-i}^\nu(b_2^*))}^{\beta, \alpha})^* I^* (\Lambda_{\nu'}(J_\nu d_1^* c_1^* J_\nu) \otimes_{tr} \Lambda_{\nu'}(J_\nu a_1 J_\nu) \otimes_{tr} \Lambda_\nu(d_2^* c_2^* a_2)) \\ &= \left[\rho_{\Lambda_\nu(\sigma_{-i}^\nu(b_2^*))}^{tr} \left(1 \otimes_{Z(M)} \sigma_{i/2}^\nu(b_1) \right) \right]^* (\Lambda_{\nu'}(J_\nu d_1^* c_1^* J_\nu) \otimes_{tr} \Lambda_{\nu'}(J_\nu a_1 J_\nu) \otimes_{tr} \Lambda_\nu(d_2^* c_2^* a_2)) \\ &= \langle d_2^* c_2^* \Lambda_\nu(a_2), \Lambda_\nu(\sigma_{-i}^\nu(b_2^*)) \rangle_{tr} \Lambda_{\nu'}(J_\nu d_1^* c_1^* J_\nu) \otimes_{tr} \sigma_{-i/2}^\nu(b_1^*) \Lambda_{\nu'}(J_\nu a_1 J_\nu) \\ &= \langle \Lambda_\nu(a_2 b_2), \Lambda_\nu(c_2 d_2) \rangle_{tr} J_\nu \Lambda_\nu(d_1^* c_1^*) \otimes_{tr} J_\nu \Lambda_\nu(a_1 b_1) \end{aligned}$$

Consequently, by definition of G :

$$G \left[\langle \Lambda_\nu(a_2 b_2), \Lambda_\nu(c_2 d_2) \rangle_{tr} J_\nu \Lambda_\nu(d_1^* c_1^*) \otimes_{tr} J_\nu \Lambda_\nu(a_1 b_1) \right]$$

is equal to the value of $G(\lambda_{\Lambda_\nu(\sigma_{i/2}^\nu(b_1)) \otimes_{tr} \Lambda_\nu(\sigma_{-i}^\nu(b_2^*))}^{\beta, \alpha})^* W^*$ on:

$$[\Lambda_{\nu'}(J_\nu a_1 J_\nu) \otimes_{tr} \Lambda_\nu(a_2)] \underset{\nu^o}{\alpha \otimes \hat{\beta}} [\Lambda_{\nu'}(J_\nu d_1^* c_1^* J_\nu) \otimes_{tr} \Lambda_\nu(d_2^* c_2^*)]$$

which is equal to the value of $(\lambda_{\Lambda_\nu(\sigma_{i/2}^\nu(d_1)) \otimes_{tr} \Lambda_\nu(\sigma_{-i}^\nu(d_2^*))}^{\beta, \alpha})^* W^*$ on:

$$[\Lambda_{\nu'}(J_\nu c_1 J_\nu) \otimes_{tr} \Lambda_\nu(c_2)] \underset{\nu^o}{\alpha \otimes \hat{\beta}} [\Lambda_{\nu'}(J_\nu b_1^* a_1^* J_\nu) \otimes_{tr} \Lambda_\nu(b_2^* a_2^*)]$$

This last vector is $\langle \Lambda_\nu(c_2 d_2), \Lambda_\nu(a_2 b_2) \rangle_{tr} J_\nu \Lambda_\nu(b_1^* a_1^*) \otimes_{tr} J_\nu \Lambda_\nu(c_1 d_1)$. Since G is closed, we get:

$$G \left[J_\nu \Lambda_\nu(d_1^* c_1^*) \otimes_{tr} J_\nu \Lambda_\nu(a_1 b_1) \right] = \left[J_\nu \Lambda_\nu(b_1^* a_1^*) \otimes_{tr} J_\nu \Lambda_\nu(c_1 d_1) \right]$$

so that G coincides with $\sigma_{tr}(F_\nu \otimes_{tr} F_\nu)$. \square

The polar decomposition of $G = ID^{1/2}$ is such that $D = \Delta_\nu^{-1} \otimes_{tr} \Delta_\nu^{-1}$ and $I = \sigma_{tr}(J_\nu \otimes_{tr} J_\nu)$ so that the scaling group is $\tau_t = \sigma_{-t}^{\nu'} \star_{Z(M)} \sigma_t^\nu$ for all $t \in \mathbb{R}$ and the unitary antipode is $R = \varsigma_{Z(M)} \circ (j \otimes_{Z(M)} j)$. We also notice that $\nu' \star_{Z(M)} \nu$ is τ -invariant.

REMARK 6.28. — If M is the commutative von Neumann algebra $L^\infty(X)$, then the structure coincides with the quantum space X .

6.7. Pairs quantum groupoid. — Let M be a von Neumann algebra. M acts on $H = L^2(M) = L_\nu^2(M)$ where ν is a n.s.f weight on M . We denote by M' the commutant of M in $\mathcal{L}(L^2(M))$. $M' \otimes M$ acts on $L^2(M) \otimes L^2(M)$.

Let α (resp. β) be the (resp. anti-) representation of M to $M' \otimes M$ such that $\alpha(m) = 1 \otimes m$ (resp. $\beta(m) = j(m) \otimes 1$) where $j(x) = J_\nu x^* J_\nu$ for all $x \in \mathcal{L}(L_\nu^2(M))$.

PROPOSITION 6.29. — *The following formula:*

$$I : [L^2(M) \otimes L^2(M)]_{\beta \otimes_\alpha} [L^2(M) \otimes L^2(M)] \rightarrow L^2(M) \otimes L^2(M) \otimes L^2(M)$$

$$[\Lambda_\nu(y) \otimes \eta]_{\beta \otimes_\alpha} \Xi \mapsto \alpha(y) \Xi \otimes \eta$$

for all $\eta \in L^2(M), \Xi \in L^2(M) \otimes L^2(M)$ and $y \in M$, defines a canonical isomorphism such that we have $I([m \otimes x]_{\beta \otimes_\alpha} [y \otimes n]) = (y \otimes mn \otimes x)I$, for all $m \in M, n \in M'$ and $x, y \in \mathcal{L}(L^2(M))$.

Proof. — Straightforward. \square

Then, we can identify $(M' \otimes M)_{\beta \star_\alpha} (M' \otimes M)$ with $M' \otimes Z(M) \otimes M$. We define a normal $*$ -homomorphism Γ by:

$$M' \otimes M \rightarrow (M' \otimes M)_{\beta \star_\alpha} (M' \otimes M)$$

$$n \otimes m \mapsto I^*(n \otimes 1 \otimes m)I = [1 \otimes m]_{\beta \otimes_\alpha} [n \otimes 1]$$

THEOREM 6.30. — $(M, M' \otimes_{Z(M)} M, \alpha, \beta, \Gamma, \nu, \nu' \otimes id, id \otimes \nu)$ is a measured quantum groupoid w.r.t ν called **pairs quantum groupoid**.

Proof. — By definition, Γ is a morphism of Hopf bimodule. We have to prove co-product relation. For all $m \in M$ and $n \in M'$, we have:

$$(\Gamma_{\beta \star_\alpha} id) \circ \Gamma(n \otimes m) = [1 \otimes m]_{\beta \otimes_\alpha} [1 \otimes 1]_{\beta \otimes_\alpha} [n \otimes 1]$$

$$= (id_{\beta \star_\alpha} \Gamma) \circ \Gamma(n \otimes m)$$

$R = \varsigma \circ (\beta_\nu \otimes \beta_\nu)$, where $\varsigma : M' \otimes M \rightarrow M \otimes M'$ is the flip, is a co-involution so it is sufficient to show that $T_L = \nu' \otimes id$ is left invariant and β -adapted w.r.t ν . Let $m \in M, n \in M'$ and $\xi \in D((L^2(M) \otimes L^2(M))_{\beta, \nu \circ})$. We put $\Phi = \nu \circ \alpha^{-1} \circ T_L$

and we compute:

$$\begin{aligned}
\omega_\xi((id \underset{\nu}{\beta} \star_\alpha \Phi) \Gamma(n \otimes m)) &= \Phi((\omega_\xi \underset{\nu}{\beta} \star_\alpha id)([1 \otimes m] \underset{\nu}{\beta} \otimes_\alpha [n \otimes 1])) \\
&= \Phi([n \otimes 1] \alpha(< [1 \otimes m] \xi, \xi >_{\beta, \nu^o})) \\
&= \nu'(n) \nu(< [1 \otimes m] \xi, \xi >_{\beta, \nu^o}) \\
&= \nu'(n) \omega_\xi(1 \otimes m) = \omega_\xi(T_L(n \otimes m))
\end{aligned}$$

Finally, we prove that $T_R = R \circ T_L \circ R = id \otimes \nu$ is α -adapted w.r.t ν . For all $t \in \mathbb{R}$, we have:

$$\sigma_t^{T_R} = \sigma_t^{\nu' \otimes \nu} |_{(M' \otimes M) \cap \beta(M)'} = \sigma_t^{\nu' \otimes \nu} |_{Z(M) \otimes M} = id \otimes \sigma_t^{\nu'} |_{Z(M) \otimes M}$$

so that we have for all $t \in \mathbb{R}$ and $m \in M$:

$$\sigma_t^{T_R} \circ \alpha(m) = 1 \otimes \sigma_t^{\nu}(m) = \alpha(\sigma_t^{\nu}(m))$$

□

REMARK 6.31. — If $M = L^\infty(X)$, we find the structure of pairs groupoid $X \times X$.

By 3.16, we can compute the pseudo-multiplicative unitary. Let first notice that $\Phi = \nu' \otimes \nu = \Psi$ so that $\lambda = \delta = 1$ and:

$$\alpha = 1 \otimes id, \hat{\alpha} = id \otimes 1, \beta = \beta_\nu \otimes 1 \text{ and } \hat{\beta} = 1 \otimes \beta_\nu$$

For example, we have $D((H \otimes H)_{\hat{\beta}, \nu^o}) \supset H \otimes D(H_{\beta_\nu}, \nu^o) = H \otimes \Lambda_\nu(\mathcal{N}_\nu)$ and for all $\eta \in H$ and $y \in \mathcal{N}_\nu$, we have $R^{\hat{\beta}, \nu^o}(\eta \otimes \Lambda_\nu(y)) = \lambda_\eta R^{\beta_\nu, \nu^o}(\Lambda_\nu(y)) = \lambda_\eta y$.

LEMMA 6.32. — We have, for all $\eta \in H$ and $e \in \mathcal{N}_\nu$:

$$I\rho_{\eta \otimes J_\nu \Lambda_\nu(e)}^{\beta, \alpha} = \lambda_\eta J_\nu e J_\nu \otimes 1 \text{ and } I\lambda_{\Lambda_\nu(y) \otimes \eta}^{\beta, \alpha} = \rho_\eta(1 \otimes y)$$

Proof. — Straightforward. □

PROPOSITION 6.33. — We have, for all $\Xi \in H \otimes H, \eta \in H$ and $m \in \mathcal{N}_\nu$:

$$W^*(\Xi \underset{\nu^o}{\alpha} \otimes_{\hat{\beta}} (\eta \otimes \Lambda_\nu(m))) = I^*(\eta \otimes (1 \otimes m) \Xi)$$

Proof. — For all $m, e \in \mathcal{N}_\nu$ and $m', e' \in \mathcal{N}_{\nu'}$, we have by the previous lemma:

$$\begin{aligned}
I\Gamma(m' \otimes m) \rho_{J_{\nu'} \Lambda_{\nu'}(e') \otimes J_\nu \Lambda_\nu(e)}^{\beta, \alpha} &= (m' \otimes 1 \otimes m) I\rho_{J_{\nu'} \Lambda_{\nu'}(e') \otimes J_\nu \Lambda_\nu(e)}^{\beta, \alpha} \\
&= (m' \otimes 1 \otimes m) \lambda_{J_{\nu'} \Lambda_{\nu'}(e')} J_\nu e J_\nu \otimes 1 \\
&= \lambda_{J_{\nu'} e' J_{\nu'} \Lambda_{\nu'}(m')} J_\nu e J_\nu \otimes m
\end{aligned}$$

On the other hand, we have by 6.29:

$$\begin{aligned}
&I([1 \otimes 1] \underset{\nu}{\beta} \otimes_\alpha [J_{\nu'} e' J_{\nu'} \otimes J_\nu e J_\nu]) W^* \rho_{\Lambda_{\nu'}(m') \otimes \Lambda_{\nu'}(m')}^{\alpha, \hat{\beta}} \\
&= (J_{\nu'} e' J_{\nu'} \otimes J_\nu e J_\nu \otimes 1) I W^* \rho_{\Lambda_{\nu'}(m') \otimes \Lambda_{\nu'}(m')}^{\alpha, \hat{\beta}}
\end{aligned}$$

Then by 3.16 and taking the limit over e and e' which go to 1, we get for all $\Xi \in H \otimes H$:

$$W^*(\Xi \underset{\nu^o}{\alpha \otimes \hat{\beta}} (\Lambda_{\nu'}(m') \otimes \Lambda_{\nu}(m))) = I^*(\Lambda_{\nu'}(m') \otimes (1 \otimes m)\Xi)$$

Now, if $\Xi \in D(\alpha(H \otimes H), \nu)$, by continuity and density of $\Lambda_{\nu'}(\mathcal{N}_{\nu'})$, we have for all $\Xi \in D(\alpha(H \otimes H), \nu)$:

$$W^*(\Xi \underset{\nu^o}{\alpha \otimes \hat{\beta}} (\eta \otimes \Lambda_{\nu}(m))) = I^*(\eta \otimes (1 \otimes m)\Xi)$$

Since $\eta \otimes \Lambda_{\nu}(m) \in D((H \otimes H)_{\hat{\beta}, \nu^o})$, the previous relation holds by continuity for all $\Xi \in H \otimes H$. \square

REMARK 6.34. — If σ denotes the flip of $L^2(M) \otimes L^2(M)$, then $\sigma \circ \hat{\beta} = \beta \circ \sigma$ and if $I' = (1 \otimes \sigma)I(\underset{\nu}{\hat{\beta} \otimes \alpha} [1 \otimes 1])\sigma_{\nu^o}$, then I' is the identification:

$$\begin{aligned} I' : [L^2(M) \otimes L^2(M)] \underset{\nu^o}{\alpha \otimes \hat{\beta}} [L^2(M) \otimes L^2(M)] &\rightarrow L^2(M) \otimes L^2(M) \otimes L^2(M) \\ \Xi \underset{\nu}{\beta \otimes \alpha} [\eta \otimes \Lambda_{\nu}(y)] &\mapsto \eta \otimes \alpha(y)\Xi \end{aligned}$$

for all $\eta \in L^2(M)$, $\Xi \in L^2(M) \otimes L^2(M)$ and $y \in M$. Consequently, by the previous proposition $W^* = I^*I'$.

COROLLARY 6.35. — *We can re-construct the underlying von Neumann algebra thanks to W :*

$$M' \otimes M = \langle (id * \omega_{\xi, \eta})(W^*) \mid \xi \in D((H \otimes H)_{\hat{\beta}, \nu^o}), \eta \in D(\alpha(H \otimes H), \nu) \rangle^{-w}$$

Proof. — By 3.23, we know that:

$$\langle (id * \omega_{\xi, \eta})(W^*) \mid \xi \in D((H \otimes H)_{\hat{\beta}, \nu^o}), \eta \in D(\alpha(H \otimes H), \nu) \rangle^{-w} \subset M' \otimes M$$

Let $\eta, \xi \in H$ and $m, e \in \mathcal{N}_{\nu}$. Then, for all $\Xi_1, \Xi_2 \in H \otimes H$, we have, by 6.32:

$$\begin{aligned} &((id * \omega_{\eta \otimes \Lambda_{\nu}(m), \xi \otimes J_{\nu} \Lambda_{\nu}(e)})(W^*)\Xi_1|\Xi_2) \\ &= (W^*(\Xi_1 \underset{\nu^o}{\alpha \otimes \hat{\beta}} \eta \otimes \Lambda_{\nu}(m))|\Xi_2 \underset{\nu}{\beta \otimes \alpha} \xi \otimes J_{\nu} \Lambda_{\nu}(e)) \\ &= (I^*(\eta \otimes (1 \otimes m)\Xi_1)|\Xi_2 \underset{\nu}{\beta \otimes \alpha} \xi \otimes J_{\nu} \Lambda_{\nu}(e)) \\ &= (\eta \otimes (1 \otimes m)\Xi_1|\xi \otimes (J_{\nu} e J_{\nu} \otimes 1)\Xi_2) \\ &= (\eta|\xi)((J_{\nu} e^* J_{\nu} \otimes m)\Xi_1|\Xi_2) \end{aligned}$$

Consequently, we get the reverse inclusion thanks to the relation:

$$(13) \quad (id * \omega_{\eta \otimes \Lambda_{\nu}(m), \xi \otimes J_{\nu} \Lambda_{\nu}(e)})(W^*) = (\eta|\xi)(J_{\nu} e^* J_{\nu} \otimes m)$$

\square

Now, we compute G so as to get the antipode.

PROPOSITION 6.36. — *If $F_\nu = S_\nu^*$ comes from Tomita's theory, we have:*

$$G = \sigma(F_\nu \otimes F_\nu)$$

Proof. — For all $a = J_\nu a_1 J_\nu \otimes a_2, b = J_\nu b_1 J_\nu \otimes b_2, c = J_\nu c_1 J_\nu \otimes c_2$ and $d = J_\nu d_1 J_\nu \otimes d_2$ be analytic elements of $M' \otimes M$ w.r.t $\nu' \otimes \nu$. Then, by 6.32, we have:

$$\begin{aligned} & (\lambda_{\Lambda_\nu(\sigma_{i/2}^\nu(b_1)) \otimes \Lambda_\nu(\sigma_{-i}^\nu(b_2^*))}^{\beta, \alpha})^* W^*(\Lambda_{\nu' \otimes \nu}(a) \underset{\nu^o}{\alpha \otimes \beta} \Lambda_{\nu' \otimes \nu}((J_\nu d_1^* J_\nu \otimes d_2^*) c^*)) \\ &= (\lambda_{\Lambda_\nu(\sigma_{i/2}^\nu(b_1)) \otimes \Lambda_\nu(\sigma_{-i}^\nu(b_2^*))}^{\beta, \alpha})^* I^*(\Lambda_{\nu'}(J_\nu d_1^* c_1^* J_\nu) \otimes (1 \otimes d_2^* c_2^*) \Lambda_{\nu' \otimes \nu}(a)) \\ &= \left[\rho_{\Lambda_\nu(\sigma_{-i}^\nu(b_2^*))} (1 \otimes \sigma_{i/2}^\nu(b_1)) \right]^* (\Lambda_{\nu'}(J_\nu d_1^* c_1^* J_\nu) \otimes (1 \otimes d_2^* c_2^*) \Lambda_{\nu' \otimes \nu}(a)) \\ &= (d_2^* c_2^* \Lambda_\nu(a_2) | \Lambda_\nu(\sigma_{-i}^\nu(b_2^*))) \Lambda_{\nu'}(J_\nu d_1^* c_1^* J_\nu) \otimes \sigma_{-i/2}^\nu(b_1^*) \Lambda_{\nu'}(J_\nu a_1 J_\nu) \\ &= \nu(d_2^* c_2^* a_2 b_2) J_\nu \Lambda_\nu(d_1^* c_1^*) \otimes J_\nu \Lambda_\nu(a_1 b_1) \end{aligned}$$

Consequently, by definition of G , we have:

$$\begin{aligned} & G[\nu(d_2^* c_2^* a_2 b_2) J_\nu \Lambda_\nu(d_1^* c_1^*) \otimes J_\nu \Lambda_\nu(a_1 b_1)] \\ &= G(\lambda_{\Lambda_\nu(\sigma_{i/2}^\nu(b_1)) \otimes \Lambda_\nu(\sigma_{-i}^\nu(b_2^*))}^{\beta, \alpha})^* W^*(\Lambda_{\nu' \otimes \nu}(a) \underset{\nu^o}{\alpha \otimes \beta} \Lambda_{\nu' \otimes \nu}((J_\nu d_1^* J_\nu \otimes d_2^*) c^*)) \\ &= (\lambda_{\Lambda_\nu(\sigma_{i/2}^\nu(d_1)) \otimes \Lambda_\nu(\sigma_{-i}^\nu(d_2^*))}^{\beta, \alpha})^* W^*(\Lambda_{\nu' \otimes \nu}(c) \underset{\nu^o}{\alpha \otimes \beta} \Lambda_{\nu' \otimes \nu}((J_\nu b_1^* J_\nu \otimes b_2^*) a^*)) \\ &= \nu(b_2^* a_2^* c_2 d_2) J_\nu \Lambda_\nu(b_1^* a_1^*) \otimes J_\nu \Lambda_\nu(c_1 d_1) \end{aligned}$$

Since G is anti-linear, we get:

$$G[J_\nu \Lambda_\nu(d_1^* c_1^*) \otimes J_\nu \Lambda_\nu(a_1 b_1)] = [J_\nu \Lambda_\nu(b_1^* a_1^*) \otimes J_\nu \Lambda_\nu(c_1 d_1)]$$

so that G coincides with $\sigma(F_\nu \otimes F_\nu)$. \square

The polar decomposition of $G = ID^{1/2}$ is such that $D = \Delta_\nu^{-1} \otimes \Delta_\nu^{-1}$ and $I = \Sigma(J_\nu \otimes J_\nu)$ so that the scaling group is $\tau_t = \sigma_{-t}^{\nu'} \otimes \sigma_t^\nu$ for all $t \in \mathbb{R}$ and the unitary antipode is $R = \varsigma \circ (\beta_\nu \otimes \beta_\nu)$. We also notice that $\nu' \otimes \nu$ is τ -invariant.

COROLLARY 6.37. — *We have $\mathcal{D}(S) = \mathcal{D}(\sigma_{i/2}^{\nu'}) \otimes \mathcal{D}(\sigma_{-i/2}^\nu)$ and we have*

$$S(J_\nu e J_\nu \otimes m^*) = J_\nu \sigma_{i/2}^\nu(m) J_\nu \otimes \sigma_{-i/2}^\nu(e^*)$$

*for all $e, m \in \mathcal{D}(\sigma_{i/2}^\nu)$. Moreover $(id * \omega_{\xi, \eta})(W) \in \mathcal{D}(S)$ and:*

$$S((id * \omega_{\xi, \eta})(W)) = (id * \omega_{\xi, \eta})(W^*)$$

for all $\xi, \eta \in D(\alpha(H \otimes H), \nu) \cap D((H \otimes H)_{\hat{\beta}}, \nu^o)$.

Proof. — The first part of the corollary is straightforward by what precedes. Let $\zeta, \eta \in H$ and $e, m \in \mathcal{D}(\sigma_{i/2}^\nu)$. By 13, we have:

$$\begin{aligned} S((id * \omega_{\zeta \otimes J_\nu \Lambda_\nu(e), \eta \otimes \Lambda_\nu(m)})(W)) &= S((\zeta|\eta)J_\nu e J_\nu \otimes m^*) \\ &= (\zeta|\eta)J\sigma_{i/2}^\nu(m)J \otimes \sigma_{-i/2}^\nu(e^*) \\ &= (id * \omega_{\zeta \otimes J_\nu \Lambda_\nu(e), \eta \otimes \Lambda_\nu(m)})(W^*) \end{aligned}$$

Since S is closed, we can conclude. \square

6.8. Operations on measured quantum groupoids. —

6.8.1. Sum of measured quantum groupoids. — A union of groupoids is still a groupoid. We establish here a similar result at the quantum level:

PROPOSITION 6.38. — *Let $(N_i, M_i, \alpha_i, \beta_i, \Gamma_i, \nu_i, T_L^i, T_R^i)_{i \in I}$ be a family of measured quantum groupoids. Then, identifying the von Neumann algebra $\bigoplus_{i \in I} M_i$ with $\bigoplus_{i \in I} M_i$ with $\bigoplus_{i \in I} M_i$ with $\bigoplus_{i \in I} M_i$, we get:*

$$\left(\bigoplus_{i \in I} N_i, \bigoplus_{i \in I} M_i, \bigoplus_{i \in I} \alpha_i, \bigoplus_{i \in I} \beta_i, \bigoplus_{i \in I} \Gamma_i, \bigoplus_{i \in I} \nu_i, \bigoplus_{i \in I} T_L^i, \bigoplus_{i \in I} T_R^i \right)$$

a measured quantum groupoid where operators act on the diagonal.

Proof. — Straightforward. \square

In particular, the sum of two quantum groups with different scaling constants ([VV03] for examples) produce a measured quantum groupoid with non scalar scaling operator.

6.8.2. Tensor product of measured quantum groupoids. — Cartesian product of groups correspond to tensor product of quantum groups. In the same way, we have:

PROPOSITION 6.39. — *Let $(N_i, M_i, \alpha_i, \beta_i, \Gamma_i, \nu_i, T_L^i, T_R^i)$ be measured quantum groupoids for $i = 1, 2$. Then, if we identify $(M_1 \otimes_{N_1} M_1) \otimes (M_2 \otimes_{N_2} M_2)$ with the von Neumann algebra $(M_1 \otimes M_2) \otimes_{N_1 \otimes N_2} (M_1 \otimes M_2)$, we have:*

$$(N_1 \otimes N_2, M_1 \otimes M_2, \alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2, \Gamma_1 \otimes \Gamma_2, \nu_1 \otimes \nu_2, T_L^1 \otimes T_L^2, T_R^1 \otimes T_R^2)$$

is a measured quantum groupoid.

Proof. — Straightforward. \square

6.8.3. *Direct integrals of measured quantum groupoids.* — In this section, X denote σ -compact, locally compact space and μ a Borel measure on X . Theory of hilbertian integrals is described in [Tak03].

PROPOSITION 6.40. — *Let $(N_p, M_p, \alpha_p, \beta_p, \Gamma_p, \nu_p, T_L^p, T_R^p)_{p \in X}$ be a family of measured quantum groupoids. If we identify the von Neumann algebras $\int_X^\oplus M_p \beta \star_\alpha M_p d\mu(p)$ and $\left(\int_X^\oplus M_p d\mu(p)\right)_{\int_X^\oplus N_p d\mu(p)}^{\beta \star_\alpha} \left(\int_X^\oplus M_p d\mu(p)\right)$, we have:*

$$\left(\int_X^\oplus N_p d\mu(p), \int_X^\oplus M_p d\mu(p), \int_X^\oplus \alpha_p d\mu(p), \int_X^\oplus \beta_p d\mu(p), \dots\right. \\ \left. \dots \int_X^\oplus \Gamma_p d\mu(p), \int_X^\oplus \nu_p d\mu(p), \int_X^\oplus T_L^p d\mu(p), \int_X^\oplus T_R^p d\mu(p)\right)$$

is a measured quantum groupoid.

Proof. — Left to the reader. □

[Bla96] gives examples. In this case, the basis is $L^\infty(X)$ and $\alpha = \beta = \hat{\beta}$. The fundamental unitary comes from a space onto the same space and then can be viewed as a field of multiplicative unitaries.

BIBLIOGRAPHY

- [ADR00] C. Anantharaman-Delaroche & J. Renault: *Amenable groupoids*, *Monographie de l'Enseignement Mathématique*, Genève (2000).
- [BS93] S. Baaj & G. Skandalis: Unitaires multiplicatifs et dualité pour les produits croisés de \mathbb{C}^* -algèbres, *Ann. Sci. ENS.*, **26** (1993), 425-488.
- [BDH88] M. Baillet, Y. Denizeau & J.F. Havet: Indice d'une espérance conditionnelle, *Compositio Math.*, **66** (1988), 199-236.
- [Bla96] E. Blanchard, Déformations de \mathbb{C}^* -algèbres de Hopf, *Bull. Soc. math. France*, **124** (1996), 141-215.
- [BNS99] G. Böhm, F. Nill & K. Szlachányi: Weak Hopf algebras I. Integral theory and \mathbb{C}^* -structure, *J. Algebra*, **221** (1999), 385-438.
- [BSz96] G. Böhm & K. Szlachányi: A coassociative \mathbb{C}^* -quantum group with non integral dimensions, *Lett. in Math. Phys.*, **35** (1996), 437-456.
- [Co79] A. Connes: Sur la théorie non commutative de l'intégration, in "Algèbres d'Opérateurs", *Lecture Notes in Mathematics*, N° **725**, pp. 19-143, Springer-Verlag, Berlin/New-York, 1979.
- [Co80] A. Connes: On the spatial theory of von Neumann algebras, *J. Funct. Analysis*, **35** (1980), 153-164.
- [Co94] A. Connes: "Non Commutative Geometry", Academic Press, 1994.
- [EN96] M. Enock & R. Nest: Inclusions of factors, multiplicative unitaries and Kac algebras, *J. Funct. Analysis*, **137** (1996), 466-543.

- [Eno00] M. Enock: Inclusions of von Neumann algebras and quantum groupoids II, *J. Funct. Analysis*, **178** (2000), 156-225.
- [Eno02] M. Enock: Quantum groupoids of compact type, to appear in *Journal de l'Institut de Mathématiques de Jussieu*.
- [Eno04] M. Enock: Inclusions of von Neumann algebras and quantum groupoids III, math.OA/0405480 (2004).
- [ES89] M. Enock & J.M Schwartz: *Kac algebras and Duality of locally compact Groups*, Springer-Verlag, Berlin, 1989.
- [EV00] M. Enock & J.M. Vallin: Inclusions of von Neumann algebras and quantum groupoids, *J. Funct. Analysis*, **172** (2000), 249-300.
- [Hah78a] P. Hahn: Haar measure for measured groupoids, *Trans. Amer. Math. soc.*, **242** (1978), 1-33.
- [Hah78b] P. Hahn: The regular representations of measured groupoids, *Trans. Amer. Math. soc.*, **242** (1978), 35-72.
- [KaV74] G.I. Kac & L. Vănnerman: Nonunimodular ring-groups and Hopf-von Neumann algebras, *Math. USSR*, **23** (1974), 185-214.
- [Kus97] J. Kustermans: KMS-weights on \mathbb{C}^* -algebras, funct-an/9704008 (1997).
- [Kus02] J. Kustermans: Induced corepresentations of locally compact quantum groups, *J. Funct. Analysis*, **194** (2002), 410-459.
- [KV99] J. Kustermans & S.Vaes: Weight theory for \mathbb{C}^* -algebraic quantum groups, preprint University College Cork & KU1 Leuven, 1999, math/99011063.
- [KV00] J. Kustermans & S.Vaes: Locally compact quantum groups, *Ann. Scient. Ec. Norm. Sup.*, **33**(6) (2000), 837-934.
- [KV03] J. Kustermans & S.Vaes: Locally compact quantum groups in the von Neumann algebraic setting, *Mathematica Scandinava*, **92**(1)(2003), 68-92.
- [KVD97] J. Kustermans & A. Van Daele: \mathbb{C}^* -algebraic quantum groups arising from algebraic quantum groups, *Int. J. Math.*, **8** (1997), 1067-1139, QA/9611023.
- [Les03] F. Lesieur: thesis, University of Orléans, available at: <http://tel.ccsd.cnrs.fr/documents/archives0/00/00/55/05>.
- [Mac66] G.W. Mackey: Ergodic theory and virtual groups, *Math. Ann.*, **186** (1966), 187-207.
- [MN91] T. Masuda & Y. Nakagami, An operator algebraic framework for the duality of quantum groups, "Mathematical Physics X", *Proc. AMP-91*, Univ. Leipzig, 1991, K. Schmüdgen, 291-295, Springer-Verlag, Berlin, 1992.
- [Nik02] D. Nikshych: On the structure of weak Hopf algebras, *Advances Math.*, **170** (2002), 257-286 in section 3.
- [Nil98] F. Nill: Axioms of weak Bialgebras, math.QA/9805104 (1998).

- [NV00] D. Nikshych & L. Vănerman: Algebraic versions of a finite dimensional quantum groupoid, *Lecture Notes in Pure and Appl. Math.*, **209** (2000), 189-221.
- [NV02] D. Nikshych & L. Vănerman: New directions in Hopf algebras, Cambridge University Press, *MSRI Publications*, **43** (2002), 211-262.
- [Ren80] J. Renault: "A groupoid approach to \mathbb{C}^* -Algebras", *Lecture Notes in Math.*, Vol.793, Springer-Verlag.
- [Ren97] J. Renault: The Fourier algebra of a measured groupoid and its multipliers, *J. Funct. Analysis*, **145** (1997), 455-490.
- [Sau83a] J.L Sauvageot: Produit tensoriel de Z -modules et applications, in Operator Algebras and their Connections with Topology and Ergodic Theory, Proceedings Busteni, Romania, *Lecture Notes in Math.*, Springer-Verlag, **1132** (1983), 468-485.
- [Sau83b] J.L Sauvageot: Sur le produit tensoriel relatif d'espaces de Hilbert, *J. Operator Theory*, **9** (1983), 237-352.
- [Sau86] J.L Sauvageot: Une relation de chaine pour les dérivées de Radon-Nykodym spatiales, *Bull. Soc. math. France*, **114** (1986), 105-117.
- [Str81] S. Strătilă: *Modular theory in operator algebras*, Abacus Press, Turnbridge Wells, England, 1981.
- [Tak03] M. Takesaki: *Theory of Operator Algebras II*, *Encyclopaedia of Mathematical Sciences*, vol. 125, Springer 2003.
- [Vae01a] S. Vaes: A Radon-Nikodym theorem for von Neumann algebras, *J. Operator Theory*, **46** (2001), 477-489.
- [Vae01b] S. Vaes: The unitary implementation of a locally compact quantum group action, *J. Funct. Analysis*, **180** (2001), 426-480.
- [Val96] J.M Vallin: Bimodules de Hopf et Poids opératoriels de Haar, *J. Operator theory*, **35** (1996), 39-65.
- [Val00] J.M Vallin: Unitaire pseudo-multiplicatif associé à un groupoïde, applications à la moyennabilité, *J. Operator theory*, **44** (2000), 347-368.
- [Val01] J.M Vallin: Groupoïdes quantiques finis, *J. Algebra*, **239** (2001), 215-261.
- [Val02] J.M Vallin: Multiplicative partial isometries and finite quantum groupoids, Proceedings of the Meeting of Theoretical Physicists and Mathematicians, Strasbourg, IRMA *Lectures in Mathematics and Theoretical Physics*, **2** (2002), 189-227.
- [Val03] J.M Vallin: Deformation of finite dimensional \mathbb{C}^* -quantum groupoids, math.QA/0310265 (2003).
- [VDa95] A. Van Daele: The Haar measure on a compact quantum group, *Proc. Amer. Math. Soc.*, **123** (1995), 3125-3128.
- [VDa98] A. Van Daele: An algebraic framework for group duality, *Adv. in Math.*, **140** (1998), 323-366.

- [VDa01] A. Van Daele: The Haar measure on some locally compact quantum groups, *math.OA/0109004* (2001).
- [VV03] S. Vaes & L. Vainerman: Extensions of locally compact quantum groups and the bicrossed product construction. *Advances in Mathematics*, **175**(1) (2003), 1-101.
- [Wor87] S.L. Woronowicz: Twisted $SU(2)$ group. An example of a non-commutative differential calculus, *Publ. RIMS*, Kyoto University, **23** (1987), 117-181.
- [Wor88] S.L. Woronowicz: Tannaka-Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ group, *Invent. Math.*, **93** (1988), 35-76.
- [Wor91] S.L. Woronowicz: Quantum $E(2)$ group and its Pontryagin dual, *Lett. Math. Phys.*, **23** (1991), 251-263.
- [Wor95] S.L. Woronowicz: Compact quantum groups: Les Houches, Session LXIV, 1995, Quantum Symmetries, Elsevier 1998.
- [Wor96] S.L. Woronowicz: From multiplicative unitaries to quantum groups, *Int. J. Math.*, **Vol. 7, N° 1** (1996), 127-149.
- [Wor01] S.L. Woronowicz: Quantum " $az + b$ " group on complex plane, *International J. Math.*, **12** (2001), 461-503.
- [WZ02] S.L. Woronowicz: Quantum " $ax + b$ " group, à paratre dans *Reviews in Mathematical Physics*.
- [Yam93] T. Yamanouchi: Duality for actions and co-actions of groupoids on von Neumann algebras, *Memoirs of the AMS*, **484** (1993).